# Deprogramming Zero in Ramanujan's Heuristic Calculus: Non-Existence as Failure of Construction, Zero-Padding as Axiom Insertion, and a Clause-Level Diagnostic for Classical Methods

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**Abstract**

Ramanujan's notebooks and the subsequent expository tradition frequently employ "zero" as a boundary value, a default coefficient, or a neutral seed in reduction procedures. This paper argues that such usage is not merely notational but reflects a systematic *completion move*.

Under an existential reading, arithmetic is a calculus of existent quantities, and there is no coherent role for a term-sign whose intended meaning is "non-existence." Operationally, the appearance of "0" in boundary/default/ empty/cancellation positions has a uniform signature: one *extends an evaluation system by new clauses* (empty-case clauses for folds, out-of-range clauses for recurrences, missing-coefficient clauses for series, cancellation-as-identity clauses), thereby *completing a partial support into a total function by axiom insertion*. We formalize this pattern as *completion-by-clause* and prove general fabrication theorems showing that seeded reduction and zero-padding are exactly such completion steps.

A central meta-result is a *depadding normalization theorem*: whenever a padded proof uses forced filler outputs only to make "out-of-support" branches contribute neutrally, the derivation can be transformed into a base derivation that restricts indices to the genuine support and eliminates the padding clauses. This upgrades "depadded forms" from an informal rewriting trick to a clause-elimination principle.

The later sections give diagnostics and illustrations drawn from families of computations commonly associated with the Ramanujan tradition (partitions, theta functions and $q$-series, continued fractions, modular equations, divergent aggregates) and from standard number-theoretic "extend-by-zero" practice. The aim is semantics-first rather than archival: citations to Berndt's editions, Andrews–Berndt, and Hardy orient to the literature rather than provide an entry-by-entry concordance. A companion toy-model application of explicit completion/depadding vocabulary (in a distinct context) appears in [11].

## Contents

**1 Introduction** **4**
1.1 A hidden convention: absence is treated as a denoting output . . . . . 4
1.2 The thesis (conditional, operational form) . . . . . 4
1.3 What this paper does (and does not do) . . . . . 4
1.4 Contributions . . . . . 4

**2 Scope and citation policy (conceptual rather than archival) 5**

**3 The logical root: a symbol cannot denote non-existence 5**
3.1 Evaluation as rule-generated denotation . . . 5
3.2 The non-existence-constant dilemma . . . 6
3.3 Why "a symbol for non-existence" cannot be a value-symbol . . . 7
3.4 Free-logic perspective: non-denoting constants are not computational values . . . 7
3.5 Identity-as-absence is the canonical contradiction . . . 8

**4 Evaluation systems: partiality, completion, and non-conservativity 8**

**5 The master pattern: completion-by-clause 8**

**6 Meta-level theorems for completion (including depadding normalization) 9**
6.1 Order-theoretic facts: extension is monotone . . . 9
6.2 Minimality: completion-by-clause is the least extension forcing the clauses . . . 9
6.3 A unification lemma: padding is seeding over empty fibers . . . 10
6.4 Determinism constraints induced by completion . . . 10
6.5 Depadding normalization (promoted flagship theorem) . . . 10

**7 Aggregation without seeding: folds as rules with no empty clause 14**
7.1 Unseeded fold . . . 14
7.2 What the mathematics actually gives you: semigroup fold vs monoid fold . . . 15
7.3 Consistency theorem: adding a seed is not innocent . . . 15

**8 Zero-padding as completion: partial support → total function by axiom insertion 16**

**9 Case study I: partitions (boundary padding) 17**
9.1 Representative modern convention (boundary padding) . . . 17
9.2 Representative identity blocks (classical; used here as illustrations) . . . 17
9.3 A finitary semantics for "unrestricted" sums in locally finite situations . . . 17
9.4 Diagnostic in clause form . . . 18

**10 Case study II: theta functions and $q$-series (coefficient padding) 18**
10.1 Representative modern convention (missing-coefficient clauses) . . . 18
10.2 A classical illustration: Jacobi triple product . . . 19
10.3 Formalization: sparse support completed to a total coefficient function . . . 19

**11 Case study III: Rogers–Ramanujan continued fraction (domain clauses and seeding) 19**
11.1 Representative modern convention (domain clauses and seeding) . . . 19
11.2 Representative block . . . 19
11.3 Diagnostic via truncations . . . 20

**12 Case study IV: modular equations and singular moduli (cancellation-as-identity) 20**
12.1 Representative modern convention (cancellation-as-identity) . . . 20
12.2 A minimal cancellation schematic . . . 20

**13 Case study V: divergent series and Ramanujan summation (explicit semantics change) 20**
13.1 Representative modern convention (explicit semantics change) . . . 20
13.2 A schematic semantics comparison . . . 20

**14 Number-theoretic implications: the same pattern outside Ramanujan 21**
14.1 Extend-by-zero is padding-by-clause . . . 21
14.2 Orthogonality and annihilation depend on the completion clause . . . 21

**15 Ramanujan-facing strengthening: 0-strict summands and locally finite "unrestricted" sums 21**
15.1 Local finiteness: when "unrestricted" sums are still finite . . . 22
15.2 Coefficient extraction as a fiberwise fold (explicitly) . . . 22
15.3 0-strict summands and strengthened depadding . . . 22
15.4 Ready-to-cite instantiations of depadding normalization . . . 26
15.4.1 Partitions: pentagonal recurrence as depadding . . . 26
15.4.2 Jacobi triple product: sparsity of squares as an empty-fiber seed . . . 26
15.4.3 Dirichlet characters: extend-by-zero as depadding . . . 26
15.5 Deprogramming as derivation normalization (explicit proof transformer) . . . 27
15.6 Ordered local finiteness (noncommutative aggregation semantics) . . . 28
15.7 Depadding as a confluent rewrite system on index lists . . . 29
15.8 Noncommutative orientation: left-extending fold-maps . . . 29
15.9 Commutation of depadding projections (intersection law) . . . 30

**16 Epistemological vs. ontological corollaries of completion 31**

**17 Conclusion 32**
17.1 Quantifying over completions (families of padded systems) . . . 33

**18 Case study VI: Ramanujan's Master Theorem (analytic continuation as domain-completion) 34**
18.1 Statement and classical context . . . 34
18.2 Operational diagnosis: three nested completion moves . . . 35
18.3 Formal clause-level structure of RMT . . . 36
18.4 Applications in Ramanujan's notebooks and their clause-level consequences . . . 37
18.4.1 Evaluation of $\int_0^\infty x^{s-1}e^{-x}\,dx$ and the Gamma function . . . 37
18.4.2 Bilateral hypergeometric series and the $\phi = \binom{2n}{n}$ instance . . . 37
18.4.3 The $\beta$-function integral and hypergeometric representations . . . 38
18.4.4 The "Ramanujan's sum" application and multiplicative structure . . . 39
18.4.5 Non-negligible consequence I: cascading completion failures . . . 39
18.4.6 Non-negligible consequence II: parametric specialisation and the failure of completion-invariance . . . 40
18.5 Depadding perspective on RMT: what is eliminable? . . . 41
18.6 Connection to mock theta functions: a second non-neutral completion . . . 41
18.7 Summary: clause-level taxonomy of completion in RMT and related results . . . 42

# 1 Introduction

## 1.1 A hidden convention: absence is treated as a denoting output

A striking feature of many classical computational presentations—including those surrounding Ramanujan—is the routine replacement of absence by a denoting object called "zero." The same move appears in many guises:

(a) **Boundary conventions** such as "$p(n) = 0$ for $n < 0$," which totalize a naturally partial support by an explicit out-of-range clause.

(b) **Coefficient padding** in $q$-series and product–sum identities where missing coefficients are completed by a rule "unmentioned index $\mapsto 0$."

(c) **Seeded reductions**, where an empty aggregation is forced to return a value (typically 0) by adding an empty-case clause.

(d) **Cancellation-as-identity**, where a collapse site is treated as producing a denoting identity element, as if cancellation itself were value-producing.

This paper takes a logic-first stance: under an existential reading of arithmetic, *there is no object corresponding to non-existence*. Accordingly, if a system produces no output at some site (empty list, out-of-range index, missing coefficient, collapse), then any practice that nevertheless returns a value there is an added clause. The goal is to name and analyze this added-clause move.

## 1.2 The thesis (conditional, operational form)

> **Thesis (conditional).** If arithmetic is read as a calculus of existent quantities, then any attempt to represent non-existence by a term-sign is logically unstable: a sign that can be computed with must be eligible to denote, hence cannot coherently denote non-existence. Therefore the appearance of "0" in boundary/default/ empty/cancellation positions is not a discovery but a completion move: one *extends the evaluation rules by new clauses* and thereby *completes a partial support into a total function by axiom insertion.*

## 1.3 What this paper does (and does not do)

We make the thesis mathematically sharp at the level of operational semantics (evaluation rules and derivability), and we isolate a master pattern—*completion-by-clause*—that unifies padding and seeding. We then prove general *meta-level* theorems about such extensions: monotonicity, minimality of completion, determinism constraints, and (centrally) a *depadding normalization* (clause-elimination) theorem.

The discussion is *not* an archival concordance that locates each displayed convention in a specific notebook entry number. References to Berndt's editions, Andrews–Berndt, and Hardy are used as orientation to the surrounding literature and style of calculation [1, 2, 3, 4, 5, 6, 7, 8, 9, 10].

## 1.4 Contributions

(i) **A logic-forced taboo (not a preference).** We show that a term-sign cannot coherently denote non-existence while functioning as a computable value.

(ii) **Fabrication theorems as non-conservative extensions.** We prove that "zero-padding" and "seeded reduction" are rule-extensions that force outputs where the base system has no output clause.

(iii) **A clause-level pattern language.** Boundary padding, coefficient padding, seeding, and cancellation-as-identity are unified as *completion-by-clause*.

(iv) **Meta-theorems for completion.** We prove monotonicity, a sharp general criterion for non-conservativity, least-completion minimality, and determinism constraints.

(v) **Depadding normalization (promoted).** We prove a clause-elimination theorem: padded derivations that use filler only neutrally can be transformed into base derivations over restricted supports.

(vi) **Illustrations across Ramanujan-adjacent families plus number-theoretic implications.** We give representative diagnostics in partitions, $q$-series, continued fractions, modular/cancellation arguments, and divergent summation; and we point out the same pattern in standard "extend-by-zero" number-theoretic practice.

# 2 Scope and citation policy (conceptual rather than archival)

This paper is semantics-first: it isolates a clause-level pattern (*completion-by-clause*, Definition 5.1) that explains several ubiquitous conventions (boundary defaults, coefficient padding, seeding of empty aggregations, and cancellation-as-identity).

**Not an entry-by-entry concordance.** Although the motivating backdrop is the Ramanujan tradition and its modern exposition (e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]), we do *not* attempt an archival audit that locates each displayed convention or identity in a specific notebook entry. Accordingly, citations in what follows function as pointers to standard references and to a style of calculation, not as claims of verbatim appearance at a unique source location.

**Interpretive discipline.** When a displayed formula is "classical" (e.g. the Jacobi triple product), it is used here as a representative site where padding/completion conventions become visible in modern practice. The point is the operational role of the added clauses, not historical priority.

**Companion usage note.** A distinct (non-Ramanujan) toy-model manuscript that explicitly applies the completion/depadding distinction to remove a $k = 0$ ambiguity in a two-series "left/right" construction is [11].

# 3 The logical root: a symbol cannot denote non-existence

## 3.1 Evaluation as rule-generated denotation

**Definition 3.1** (Denoting domain and evaluation judgement)**.** Let $\mathcal{Q}$ be a nonempty set (the domain of *existent quantities*) and Term a set of closed term-signs. An evaluation system $\mathcal{E}$ is a rule set generating judgements

$$\mathcal{E} \vdash t \Downarrow q \qquad (t \in \text{Term},\ q \in \mathcal{Q}).$$

If there is no $q \in \mathcal{Q}$ with $\mathcal{E} \vdash t \Downarrow q$, we say evaluation *fails* at $t$ in $\mathcal{E}$.

**Convention 3.2** (Value-literals in the term language)**.** We assume the term language contains *value literals*: for each $q \in \mathcal{Q}$ there is a closed term (also written $q$) intended to denote that value.

Every evaluation system considered in the paper includes the literal clause

$$\mathcal{E} \vdash q \Downarrow q \qquad (q \in \mathcal{Q}).$$

This convention avoids a persistent object-language/meta-language mismatch when we define fold terms over lists of "values."

**Convention 3.3** (Extensionality of value judgements / equality-respects-evaluation)**.** The right-hand component of an evaluation judgement $t \Downarrow q$ is an element of the meta-level set $\mathcal{Q}$. Accordingly, evaluation judgements are extensional in the value: if $q, q' \in \mathcal{Q}$ and $q = q'$ as elements of $\mathcal{Q}$, then the judgements $t \Downarrow q$ and $t \Downarrow q'$ are the same judgement. Equivalently, we freely rewrite the displayed value in an evaluation judgement using equalities in $\mathcal{Q}$.

**Convention 3.4** (Canonical enumeration of finite fibers into lists)**.** For each ambient set $W$, we fix once and for all a choice of a *canonical enumeration* function

$$\mathsf{enum}_W : \{\text{finite subsets of } W\} \to \{\text{finite lists in } W\},$$

such that $\mathsf{enum}_W(F)$ lists each element of $F$ exactly once (in some fixed but arbitrary order determined by the choice function). When $W$ is clear from context, we write $\mathsf{enum}$ for $\mathsf{enum}_W$.

Thus, when a definition says "fold over the fiber $R[d]$," it abbreviates "fold over the list $\mathsf{enum}(R[d])$."

When the aggregation operation is commutative/associative (e.g. addition in a commutative monoid), results do not depend on the particular choice of $\mathsf{enum}$; otherwise, the choice is treated as part of the syntax/semantics.

**Convention 3.5** (List literals are syntactic even when evaluation fails)**.** We assume the term language contains list-literals for finite lists (including the empty list $[]$), so that terms like $\mathsf{Fold}([])$ and $\mathsf{FoldMap}(g, [])$ are *old terms* even if the base system has no evaluation clause for them. Accordingly, "empty-case evaluation" is treated as a genuine completion-by-clause step.

**Convention 3.6** (Natural numbers and signed computation)**.** Throughout, we take

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

When an example involves alternating signs or subtraction (e.g. the pentagonal recurrence for partitions), we implicitly work in a denoting domain $\mathcal{Q}$ that supports the required algebra (at least an abelian group under addition), typically $\mathcal{Q} = \mathbb{Z}$, while values such as $p(n)$ may land in $\mathbb{N} \subseteq \mathbb{Z}$.

**Remark 3.7.** This paper treats "undefined" or "no value" operationally: *failure of construction* means there is no derivable judgement $t \Downarrow q$. No extra term-sign is introduced to *name* the failure *inside* the same denoting domain.

### 3.2 The non-existence-constant dilemma

**Theorem 3.8** (Non-existence constant dilemma)**.** *Fix $\mathcal{E}$,* Term*, and $\mathcal{Q}$ as in Definition 3.1. Let $c \in$* Term *be introduced with the intended meaning "non-existence."*

*Then there are only two coherent possibilities:*

*(i) (*Reification*) There exists $q \in \mathcal{Q}$ such that $\mathcal{E} \vdash c \Downarrow q$. Then $c$ denotes an existent quantity and therefore cannot denote non-existence.*

*(ii) (*Judgemental non-denotation*) For every $q \in \mathcal{Q}$ one has $\mathcal{E} \nvdash c \Downarrow q$. Then $c$ is not a computable* value *of the calculus, and any practice that nevertheless treats occurrences of $c$ as if they supply usable outputs must (operationally) either:*

*(a) add new evaluation clauses producing some designated filler value $e \in \mathcal{Q}$ at the previously failing sites, or*

*(b) explicitly extend the domain (e.g. $\mathcal{Q}_\perp = \mathcal{Q} \cup \{\perp\}$) and add rules evaluating failing sites to $\perp$.*

*Therefore no term-sign can coherently denote non-existence* and *function as a value in the same denoting calculus.*

*Proof.* Either some evaluation judgement for $c$ is derivable in $\mathcal{E}$, or none is. In the first case $c$ denotes an element of $\mathcal{Q}$. In the second case, treating the failing site as value-producing requires adding evaluation rules (possibly after a domain extension). □

### 3.3 Why "a symbol for non-existence" cannot be a value-symbol

Theorem 3.8 is elementary, but it is often resisted because one silently shifts between two distinct roles:

(i) a *value-term* (eligible to be the output of evaluation), and

(ii) a *failure marker* (a meta-level report that evaluation did not produce a value).

The core point is that (ii) is not an element of the object domain $\mathcal{Q}$ unless one explicitly changes the domain.

**Remark 3.9** (Set-theoretic "0 as empty set" still does not give non-existence)**.** In ZF-style foundations one may define $0 := \varnothing$. But $\varnothing$ is not "non-existence"; it is a perfectly existent object (a set) characterized by a property. So even in that regime, using 0 does not encode non-existence; it encodes a particular existent object chosen as a boundary/identity witness.

### 3.4 Free-logic perspective: non-denoting constants are not computational values

One can attempt to evade Theorem 3.8 by adopting a *free logic* or a *partial-term* semantics, in which some closed terms fail to denote. But then such terms cannot play the same role as ordinary values without additional machinery: one needs a definedness predicate (or evaluation preconditions), and substitution/equational reasoning must be restricted to defined terms.

Operationally, this is exactly the paper's point: "absence" is not a value in $\mathcal{Q}$; it is a failure state that lives at the level of judgements (no derivation) unless one performs an explicit completion (adjoining a new element such as $\perp$ and extending the rules). Renaming the failure does not eliminate the completion step; it only changes its packaging.

**Definition 3.10** (Two honest ways to represent failure)**.** There are two coherent ways to represent "no value":

(i) **Judgemental failure (strict):** keep the same domain $\mathcal{Q}$ and let failure mean "no judgement $t \Downarrow q$ is derivable."

(ii) **Domain extension (explicit):** change the domain to $\mathcal{Q}_\perp := \mathcal{Q} \cup \{\perp\}$ and extend the evaluation rules so that some terms evaluate to $\perp$.

In case (ii), $\perp$ is not "non-existence" inside $\mathcal{Q}$; it is a new existent marker in an enlarged domain.

**Remark 3.11.** Most informal uses of "0 means nothing" oscillate between (i) and (ii) while writing as if neither change occurred. The present paper's diagnostic insists that any such move is either judgemental failure (no value) or a rule/domain extension.

**Remark 3.12** (Why "$\perp$"/"undefined" markers do not solve the problem)**.** Any proposed replacement token for 0 intended to mean non-existence falls under the same dilemma: if it evaluates, it reifies; if it does not, any use of it as an output is an inserted clause. So the critique cannot be solved by renaming the nothing.

### 3.5 Identity-as-absence is the canonical contradiction

**Proposition 3.13** (Identity used as absence is a category error)**.** *Suppose a term-sign* 0 *is treated both as:*

*(i) an identity element in value-returning computations (i.e. it is a legitimate output in $\mathcal{Q}$), and*

*(ii) the representation of absence/non-existence in collapse/empty/out-of-range cases.*

*Then the notation forces a category error: it assigns a denoting element of $\mathcal{Q}$ to do the work of what, by intention, is not an element of $\mathcal{Q}$.*

## 4 Evaluation systems: partiality, completion, and non-conservativity

**Definition 4.1** (Base vs. completed systems)**.** A *base* evaluation system $\mathcal{E}_0$ contains only clauses reflecting the intended construction/support. A *completion* of $\mathcal{E}_0$ is an extension $\mathcal{E}_0 \subseteq \mathcal{E}_1$ that adds clauses forcing evaluation at sites where $\mathcal{E}_0$ fails to derive any value.

**Definition 4.2** (Conservative vs. non-conservative extension)**.** An extension $\mathcal{E}_0 \subseteq \mathcal{E}_1$ is *conservative* (over the old terms) if for every old term $t$ and old value $q$,

$$\mathcal{E}_0 \vdash t \Downarrow q \iff \mathcal{E}_1 \vdash t \Downarrow q.$$

It is *non-conservative* if it creates new value-judgements for some old term and old value.

**Remark 4.3.** Padding and seeding are typically non-conservative: they create value-judgements at previously failing sites. This is not "bad" by itself; it is simply a semantics change. Under the existential reading, the change is exactly the fabrication move.

## 5 The master pattern: completion-by-clause

**Definition 5.1** (Completion-by-clause)**.** A *completion-by-clause* is an extension $\mathcal{E}_0 \subseteq \mathcal{E}_0 + \Delta$ such that $\Delta$ consists of clauses of one of the following types:

(a) (*Empty-case clauses*) forcing an output for an empty aggregation;

(b) (*Out-of-range clauses*) forcing an output for inputs outside a designated support;

(c) (*Missing-coefficient clauses*) forcing a coefficient output for indices not constructed by a sparse representation;

(d) (*Cancellation-as-identity clauses*) forcing an identity output at a collapse/cancellation site.

**Proposition 5.2** (Why "0" is the canonical completion witness)**.** *In standard practice the filler output used in the added clauses $\Delta$ is often an additive identity and is written* 0*. Operationally, this is a design choice:* 0 *is the conventional* value chosen *to be inserted by the completion clauses.*

# 6 Meta-level theorems for completion (including depadding normalization)

This section isolates general theorems *about* evaluation systems and completion-by-clause extensions. The flagship result is Theorem 6.18 (depadded normalization / clause-elimination).

## 6.1 Order-theoretic facts: extension is monotone

**Proposition 6.1** (Monotonicity of derivability under extension)**.** *If $\mathcal{E}_0 \subseteq \mathcal{E}_1$, then for any $t \in$* Term *and $q \in \mathcal{Q}$,*

$$\mathcal{E}_0 \vdash t \Downarrow q \implies \mathcal{E}_1 \vdash t \Downarrow q.$$

*Proof.* Any derivation tree using only rules in $\mathcal{E}_0$ is also a derivation tree in $\mathcal{E}_1$. □

**Proposition 6.2** (General criterion for non-conservativity)**.** *Let $\mathcal{E}_0 \subseteq \mathcal{E}_1$ be an extension over the same* Term, $\mathcal{Q}$*. Then the extension is non-conservative (Definition 4.2) iff there exist an* old *term $t$ and an* old *value $q \in \mathcal{Q}$ such that*

$$\mathcal{E}_1 \vdash t \Downarrow q \quad \textit{and} \quad \mathcal{E}_0 \nvdash t \Downarrow q.$$

*Proof.* Unpack Definition 4.2. □

**Definition 6.3** (Value set and failure-filling extension)**.** For an evaluation system $\mathcal{E}$, define the *value set* of a (closed) term $t$ by

$$\mathrm{Vals}_{\mathcal{E}}(t) := \{\, q \in \mathcal{Q} : \ \mathcal{E} \vdash t \Downarrow q \,\}.$$

An extension $\mathcal{E}_0 \subseteq \mathcal{E}_1$ is *failure-filling over old terms* if for every *old* term $t$, either

$$\mathrm{Vals}_{\mathcal{E}_1}(t) = \mathrm{Vals}_{\mathcal{E}_0}(t),$$

or

$$\mathrm{Vals}_{\mathcal{E}_0}(t) = \varnothing \ \text{ and } \ \mathrm{Vals}_{\mathcal{E}_1}(t) \neq \varnothing.$$

Equivalently: the only old terms for which $\mathcal{E}_1$ creates new value judgements are those at which $\mathcal{E}_0$ had total failure.

**Corollary 6.4** (Non-conservativity by filling a prior failure)**.** *If $\mathcal{E}_0 \subseteq \mathcal{E}_1$ is failure-filling (Definition 6.3), then it is non-conservative iff there exists an old term $t$ such that*

$$\Big(\forall q \in \mathcal{Q} \ \ \mathcal{E}_0 \nvdash t \Downarrow q\Big) \quad \textit{and} \quad \Big(\exists q \in \mathcal{Q} \ \ \mathcal{E}_1 \vdash t \Downarrow q\Big).$$

*Proof.* Under failure-filling, a new judgement for an old term can only appear at a term where $\mathcal{E}_0$ had no judgement at all; now apply Proposition 6.2. □

## 6.2 Minimality: completion-by-clause is the least extension forcing the clauses

**Proposition 6.5** (Least completion property)**.** *Fix $\mathcal{E}_0$ and a set of new clauses $\Delta$. Let $\mathcal{E}^* := \mathcal{E}_0 + \Delta$. Then $\mathcal{E}^*$ is the least extension of $\mathcal{E}_0$ that validates all clauses in $\Delta$: for any evaluation system $\mathcal{E}$,*

$$\mathcal{E} \supseteq \mathcal{E}_0 \ \ \textit{and} \ \ \mathcal{E} \textit{ contains every clause in } \Delta \quad \Rightarrow \quad \mathcal{E} \supseteq \mathcal{E}^*.$$

*Proof.* Immediate from the definition of union-of-rule-sets. □

### 6.3 A unification lemma: padding is seeding over empty fibers

**Definition 6.6** (Fiberwise aggregation schema)**.** Let $D$ be an index set and let $R \subseteq D \times W$ be a relation of "witnesses" $w \in W$ contributing to index $d \in D$. Let $v : W \to \mathcal{Q}$ be a value map, and let $\mathsf{Fold}$ aggregate finite lists of term-signs whose evaluations land in $\mathcal{Q}$. Define the *fiber* at $d$ by $R[d] := \{w \in W : (d, w) \in R\}$.

Assume $R[d]$ is finite for the $d$ under consideration, so that by Convention 3.4 we have a finite witness list $\mathsf{enum}(R[d]) = [w_1, \ldots, w_n]$. A *fiberwise definition* of an output $F(d)$ is the fold over the corresponding value-literals:

$$F(d) \text{ is defined by folding the list } [\, v(w_1), \ldots, v(w_n)\,].$$

(Here we use Convention 3.2 to regard each $v(w_i) \in \mathcal{Q}$ also as a closed term.)

**Proposition 6.7** (Padding equals a seed axiom for empty fibers)**.** *In the fiberwise schema (Definition 6.6), insisting that $F(d)$ return a fixed value $e \in \mathcal{Q}$ when $R[d] = \varnothing$ is exactly an* empty-case clause *for the fold. Equivalently, "padding by $e$" is "seeding by $e$" applied fiberwise.*

**Remark 6.8.** This is the precise sense in which coefficient padding in $q$-series is not a separate phenomenon from "empty sum equals 0": coefficient extraction creates empty fibers for missing exponents, and the convention that these coefficients are 0 is the seeded empty-case clause.

**Remark 6.9** (Order dependence vs. commutativity)**.** Because folds are defined on lists, aggregating a finite *set* of witnesses requires an enumeration choice (Convention 3.4). When $\oplus$ is commutative/associative (as in ordinary addition), the aggregated value is independent of the choice; otherwise the enumeration is part of the semantics.

### 6.4 Determinism constraints induced by completion

**Definition 6.10** (Determinism on a class of terms)**.** Let $\mathcal{E}$ be an evaluation system. For a set of closed terms $X \subseteq \mathrm{Term}$, we say $\mathcal{E}$ *is deterministic on* $X$ if for every $t \in X$ the value set has cardinality at most 1:

$$|\mathrm{Vals}_{\mathcal{E}}(t)| \leq 1.$$

Equivalently: if $\mathcal{E} \vdash t \Downarrow q$ and $\mathcal{E} \vdash t \Downarrow q'$, then $q = q'$.

**Proposition 6.11** (Completion + determinism forces identifications)**.** *Let $\mathcal{E}_1 = \mathcal{E}_0 + \Delta$ be an extension. Assume $\mathcal{E}_1$ is* deterministic *on old terms: for every old $t$, if $\mathcal{E}_1 \vdash t \Downarrow q$ and $\mathcal{E}_1 \vdash t \Downarrow q'$, then $q = q'$. If $\mathcal{E}_0 \vdash t \Downarrow q$ for some old $t$ but $\Delta$ enables a derivation of $t \Downarrow q'$ along a new path, then determinism forces $q = q'$.*

**Remark 6.12.** Theorem 7.7 is the fold-specialization of this principle.

### 6.5 Depadding normalization (promoted flagship theorem)

To state a general depadding theorem, we isolate the structural situation that occurs in boundary padding and coefficient padding: padding clauses generate *filler values* that are neutral under the surrounding aggregation, so padded branches can be erased.

**Definition 6.13** (Seeded fold-map (right-extending / left fold with seed))**.** Let $\oplus$ be a binary operation on $\mathcal{Q}$ and let $e \in \mathcal{Q}$ be a chosen element.

Fix an index set $D$ and a term-former $g(d) \in \mathrm{Term}$ $(d \in D)$. The *seeded fold-map term-former*

$$\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$$

takes a finite list $L = [d_1, \ldots, d_n]$ in $D$ and denotes the seeded *left fold* in the accumulator-on-the-left sense:

$$(\cdots((e \oplus a_1) \oplus a_2)\cdots) \oplus a_n,$$

where each $a_i$ is the value of $g(d_i)$, with the understanding that for $n = 0$ it returns $e$.

Operationally, we assume the evaluation system contains fold-map clauses:

(i) **Empty:** $\mathsf{FoldMap}_{\oplus,e}(g, []) \Downarrow e$.

(ii) **Step (right-extension):** from $\mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow x$ and $g(d) \Downarrow a$, infer

$$\mathsf{FoldMap}_{\oplus,e}(g, L +\!\!+ [d]) \Downarrow x \oplus a.$$

**Definition 6.14** (Fold-map syntax-directedness)**.** Fix $(\oplus, e, g)$ as in Definition 6.13. An evaluation system $\mathcal{E}$ is said to be $\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$*-syntax-directed* if the *only* clauses in $\mathcal{E}$ whose conclusion has the form

$$\mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$$

are exactly the Empty/Step clauses of Definition 6.13. Equivalently: every derivation of such a judgement ends with Empty when $L = []$ and ends with Step when $L = L_0 +\!\!+ [d]$.

**Definition 6.15** (Axiomatic atomicity of a term-former)**.** Let $\mathcal{E}$ be an evaluation system and let $g(d) \in \mathrm{Term}$ be a term-former indexed by $d \in D$. We say that $g$ *is axiomatic-atomic in* $\mathcal{E}$ if every clause in $\mathcal{E}$ whose conclusion has the form

$$g(d) \Downarrow q$$

is an *axiom clause*, i.e. it has no premises.

Equivalently: $\mathcal{E}$ contains some set of axioms $g(d) \Downarrow q$, and there are *no* inference rules with premises that conclude a judgement of the form $g(d) \Downarrow q$.

**Lemma 6.16** (Prefix extraction for seeded fold-map)**.** *Assume $\mathcal{E}$ is $\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$-syntax-directed (Definition 6.14). Let $L = [d_1, \ldots, d_n]$ be a finite list in $D$. If $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$, then for each $j = 0, 1, \ldots, n$ there exists $q_j \in \mathcal{Q}$ such that*

$$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, [d_1, \ldots, d_j]) \Downarrow q_j.$$

*Proof.* Induct on $n$.

If $n = 0$, then $L = []$ and the Empty clause yields the required derivation for $j = 0$.

If $n > 0$, write $L = L_0 +\!\!+ [d_n]$ with $L_0 = [d_1, \ldots, d_{n-1}]$. Since $\mathcal{E}$ is syntax-directed, any derivation of $\mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$ ends with Step and therefore contains a premise derivation of $\mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow q_{n-1}$ for some $q_{n-1}$. Apply the induction hypothesis to that premise derivation to obtain derivations for all prefixes of $L_0$, and keep the given derivation for the full list. □

**Lemma 6.17** (Erasing neutral fold-map inputs (forced-neutral case))**.** *Assume:*

*(i) $\mathcal{E}$ is $\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$-syntax-directed (Definition 6.14),*

*(ii) $e$ is a* right identity *for $\oplus$, i.e. $x \oplus e = x$ for all $x \in \mathcal{Q}$,*

*(iii)* $d \in D$ *satisfies the* forced-neutral *condition*

$$\mathrm{Vals}_{\mathcal{E}}(g(d)) = \{e\}.$$

*Let* $L$ *be a finite list in* $D$ *that contains an occurrence of* $d$*, and let* $L'$ *be the list obtained from* $L$ *by deleting* one occurrence *of* $d$ *(preserving the order of the remaining entries). Then for every* $q \in \mathcal{Q}$*,*

$$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L') \Downarrow q.$$

*Proof.* We prove both implications simultaneously by induction on the length of $L$.

Write $L = L_0 +\!\!+ [d_0]$. There are two cases depending on whether the deleted occurrence is the final entry.

- **Deleted occurrence is not last.** Then $d_0$ is not deleted, so $L'$ has the form $L_0' +\!\!+ [d_0]$ where $L_0'$ is obtained from $L_0$ by deleting one occurrence of $d$.

  ($\Rightarrow$) Suppose $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$. By syntax-directedness, the last rule is Step, so there exist $x, a \in \mathcal{Q}$ with

  $$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow x, \qquad \mathcal{E} \vdash g(d_0) \Downarrow a, \qquad q = x \oplus a.$$

  By the induction hypothesis applied to $L_0$ and $L_0'$, we get $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0') \Downarrow x$. Reapplying Step with the unchanged premise derivation of $g(d_0) \Downarrow a$ yields

  $$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0' +\!\!+ [d_0]) \Downarrow x \oplus a = q,$$

  i.e. $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L') \Downarrow q$.

  ($\Leftarrow$) The reverse direction is symmetric: start from a derivation of $\mathsf{FoldMap}_{\oplus,e}(g, L') = \mathsf{FoldMap}_{\oplus,e}(g, L_0' +\!\!+ [d_0])$ and use the induction hypothesis to replace the premise $\mathsf{FoldMap}_{\oplus,e}(g, L_0') \Downarrow x$ by $\mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow x$, then reapply Step.

- **Deleted occurrence is last.** Then $d_0 = d$ and $L' = L_0$.

  ($\Rightarrow$) Suppose $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0 +\!\!+ [d]) \Downarrow q$. By syntax-directedness, the last rule is Step, so for some $x, a \in \mathcal{Q}$,

  $$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow x, \qquad \mathcal{E} \vdash g(d) \Downarrow a, \qquad q = x \oplus a.$$

  By forced-neutrality, $\mathrm{Vals}_{\mathcal{E}}(g(d)) = \{e\}$, hence $a = e$, so $q = x \oplus e = x$ by right-identity. Thus $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow q$, i.e. $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L') \Downarrow q$.

  ($\Leftarrow$) Suppose $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow q$. Also, by forced-neutrality, $\mathcal{E} \vdash g(d) \Downarrow e$. Applying Step yields

  $$\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L_0 +\!\!+ [d]) \Downarrow q \oplus e.$$

  By right-identity $q \oplus e = q$, and by extensionality of value judgements (Convention 3.3) this is $\mathcal{E} \vdash \mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$.

□

**Theorem 6.18** (Depadding normalization / clause-elimination)**.** *Let* $\mathcal{E}_0$ *be a base evaluation system. Fix:*

*(i) an index set* $D$ *and designated support* $S \subseteq D$*,*

*(ii) a term-former $g(d) \in \mathrm{Term}$ $(d \in D)$ such that evaluation* fails *outside $S$ in $\mathcal{E}_0$, i.e.*

$$\forall d \in D \setminus S \quad \forall q \in \mathcal{Q} \quad \mathcal{E}_0 \nvdash g(d) \Downarrow q,$$

*(iii) a chosen filler value $e \in \mathcal{Q}$,*

*(iv) an extension $\mathcal{E}_1 = \mathcal{E}_0 + \Delta$ where every clause in $\Delta$ is a padding clause of the form*

$$g(d) \Downarrow e \qquad (d \in D \setminus S),$$

*(v) **(Axiomatic atomicity / no derived on-support values.)** Both $\mathcal{E}_0$ and $\mathcal{E}_1$ are axiomatic-atomic for $g$ in the sense of Definition 6.15. Moreover:*

*(a) for each $s \in S$, the set of axioms in $\mathcal{E}_1$ with conclusion $g(s) \Downarrow q$ is* exactly the same *as in $\mathcal{E}_0$ (so $\mathcal{E}_1$ introduces no new $g(s)$-axioms on support), and*

*(b) for each $d \in D \setminus S$, the* only *$g(d)$-axiom in $\mathcal{E}_1$ is the padding axiom $g(d) \Downarrow e$ (so $\mathrm{Vals}_{\mathcal{E}_1}(g(d)) = \{e\}$ off support).*

*(vi) a binary operation $\oplus$ on $\mathcal{Q}$ for which $e$ is a right identity: $x \oplus e = x$ for all $x \in \mathcal{Q}$,*

*(vii) the fold-map term-former $\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$ with Empty/Step clauses present already in $\mathcal{E}_0$ (hence also in $\mathcal{E}_1$); see Definition 6.13,*

*(viii) the evaluation system $\mathcal{E}_1$ is $\mathsf{FoldMap}_{\oplus,e}(g, \cdot)$-syntax-directed (Definition 6.14).*

*For a finite list $L$ in $D$, let $L|_S$ be the sublist obtained by deleting all entries not in $S$ (preserving order). Define the term*

$$T(L) := \mathsf{FoldMap}_{\oplus,e}(g, L).$$

*Then, for every finite list $L$ and every $q \in \mathcal{Q}$,*

$$\mathcal{E}_1 \vdash T(L) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E}_0 \vdash T(L|_S) \Downarrow q.$$

*Moreover, from any derivation of $T(L) \Downarrow q$ in $\mathcal{E}_1$, one can mechanically construct a derivation of $T(L|_S) \Downarrow q$ in $\mathcal{E}_0$ by first repeatedly applying Lemma 6.17 to erase every occurrence of an index $d \notin S$ from the fold-map list (using the forced-neutral fact $\mathrm{Vals}_{\mathcal{E}_1}(g(d)) = \{e\}$), and then reading the resulting derivation as a derivation in $\mathcal{E}_0$ (using hypothesis (v) to eliminate any remaining dependence on padding clauses).*

*Proof.* ($\Rightarrow$) Consider a derivation of $T(L) \Downarrow q$ in $\mathcal{E}_1$, i.e. $\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$.

For any $d \notin S$, hypothesis (v) implies $\mathrm{Vals}_{\mathcal{E}_1}(g(d)) = \{e\}$. Since $e$ is a right identity for $\oplus$, Lemma 6.17 applies and allows us to delete *one occurrence at a time* of any such $d$ from the fold-map input list without changing derivability of the output value. Iterating over all occurrences of indices outside $S$ yields

$$\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,e}(g, L|_S) \Downarrow q.$$

Now every index in $L|_S$ lies in $S$. In the erased derivation, every leaf judgement of the form $g(s) \Downarrow a$ (with $s \in S$) must be an axiom clause (axiomatic atomicity). By hypothesis (v)(a), these on-support $g(s)$-axioms are already axioms of $\mathcal{E}_0$. Since the fold-map rules are present in $\mathcal{E}_0$ as well, the entire erased derivation is a derivation in $\mathcal{E}_0$:

$$\mathcal{E}_0 \vdash T(L|_S) \Downarrow q.$$

($\Leftarrow$) We prove by induction on the length of $L$ that

$$\mathcal{E}_0 \vdash T(L|_S) \Downarrow q \implies \mathcal{E}_1 \vdash T(L) \Downarrow q.$$

The case $L = []$ is immediate from the Empty clause, which is present already in $\mathcal{E}_0 \subseteq \mathcal{E}_1$.

Assume $L = L_0 +\!\!+ [d]$ and $\mathcal{E}_0 \vdash T((L_0 +\!\!+ [d])|_S) \Downarrow q$.

- If $d \notin S$, then $(L_0 +\!\!+ [d])|_S = L_0|_S$. By the induction hypothesis, $\mathcal{E}_1 \vdash T(L_0) \Downarrow q$. Also $\mathcal{E}_1 \vdash g(d) \Downarrow e$ by padding, so applying Step gives

  $$\mathcal{E}_1 \vdash T(L) \Downarrow (q \oplus e).$$

  By the right-identity law in $\mathcal{Q}$, $q \oplus e = q$, so by extensionality of value judgements (Convention 3.3) this is exactly $\mathcal{E}_1 \vdash T(L) \Downarrow q$.

- If $d \in S$, then $(L_0 +\!\!+ [d])|_S = (L_0|_S) +\!\!+ [d]$. Since $\mathcal{E}_0 \subseteq \mathcal{E}_1$ and $\mathcal{E}_1$ is FoldMap-syntax-directed, $\mathcal{E}_0$ has no other FoldMap-concluding rules besides Empty/Step; thus the last step of the $\mathcal{E}_0$-derivation must be Step. Therefore there exist $x, a \in \mathcal{Q}$ such that

  $$\mathcal{E}_0 \vdash T(L_0|_S) \Downarrow x, \qquad \mathcal{E}_0 \vdash g(d) \Downarrow a, \qquad q = x \oplus a.$$

  By the induction hypothesis applied to $L_0$ and value $x$, we obtain $\mathcal{E}_1 \vdash T(L_0) \Downarrow x$. Also $\mathcal{E}_1 \vdash g(d) \Downarrow a$ by monotonicity (Proposition 6.1). Applying Step in $\mathcal{E}_1$ yields $T(L) \Downarrow x \oplus a = q$.

This completes the induction and hence the ($\Leftarrow$) direction. □

**Remark 6.19** (Why only right identity is needed here)**.** Because $\mathsf{FoldMap}_{\oplus,e}$ is defined by right-extending Step clauses (Definition 6.13), erasing or inserting a padded contribution uses only $x \oplus e = x$ at the moment that padded contribution is folded in. No associativity or left-identity is required for the clause-elimination argument in this right-extension scheme.

# 7 Aggregation without seeding: folds as rules with no empty clause

## 7.1 Unseeded fold

**Definition 7.1** (Aggregation data)**.** An aggregation datum is a relation

$$\mathrm{Add} \subseteq \mathcal{Q} \times \mathcal{Q} \times \mathcal{Q},$$

read $\mathrm{Add}(a, b, c)$: "$c$ is a permissible aggregate of $a$ and $b$."

**Definition 7.2** (Unseeded fold system)**.** Fix Add. Define a fold-judgement $\mathsf{Fold}_{\mathrm{Add}}(L) \Downarrow q$ where $L$ is a finite list of *closed terms* in Term. The unseeded fold evaluation system $\mathcal{F}(\mathrm{Add})$ is the rule set consisting *exactly* of:

(i) **Singleton:** from $t \Downarrow a$ infer $\mathsf{Fold}_{\mathrm{Add}}([t]) \Downarrow a$.

(ii) **Step:** from $\mathsf{Fold}_{\mathrm{Add}}(L) \Downarrow x$, $t \Downarrow a$, and $\mathrm{Add}(x, a, y)$ infer $\mathsf{Fold}_{\mathrm{Add}}(L +\!\!+ [t]) \Downarrow y$.

No empty-list clause is included.

**Definition 7.3** (Seed axiom)**.** A seed axiom is a clause $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow e$ for some chosen $e \in \mathcal{Q}$ (typically written 0).

**Theorem 7.4** (Seeding is completion-by-clause)**.** *Adding a seed axiom to* $\mathcal{F}(\mathrm{Add})$ *is completion-by-clause (Definition 5.1): it extends the evaluation rules by a new clause that forces an output at the empty-case site, hence is non-conservative.*

*Proof.* In $\mathcal{F}(\mathrm{Add})$, every derivation starts from Singleton, and Step preserves nonemptiness of the list subject. Therefore no judgement $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow q$ is derivable in the base system. By Convention 3.5, the term $\mathsf{Fold}_{\mathrm{Add}}([])$ is already an old term; adding $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow e$ creates a new derivable judgement for an old term and old value, so the extension is non-conservative by Proposition 6.2. □

**Example 7.5** (Why the empty case is not "already 0")**.** Even when Add comes from addition in a ring and there is an identity $0 \in \mathcal{Q}$, the unseeded system still does not derive $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow 0$. That derivation requires the extra empty-case clause. So the seed is not a theorem; it is a rule.

## 7.2 What the mathematics actually gives you: semigroup fold vs monoid fold

The unseeded fold system $\mathcal{F}(\mathrm{Add})$ corresponds to the fact that the recursive specification of an aggregation over lists has *no clause for the empty list* unless one adds it. Put differently: the induction/recursion principle for *nonempty* lists is strictly weaker than the principle for all lists.

This aligns with standard algebraic structure, with one important qualification about what is assumed vs. what is derived:

(i) If $\oplus$ is associative, then an associative binary operation without a specified empty-case value corresponds to a *semigroup* (and a fold over nonempty lists is well-behaved without any unit).

(ii) Specifying an empty-case output (a seed) is extra structure: it singles out a distinguished element $e$ intended to behave as a unit-like value for the particular fold scheme being used. In a left-fold scheme, determinism forces $e$ to be a *left identity* (Theorem 7.7). If one additionally assumes associativity and the corresponding right-identity law, one recovers the usual notion of a *monoid*.

The crucial point is that any such identity element is *extra structure*; it is not determined by the binary operation alone.

## 7.3 Consistency theorem: adding a seed is not innocent

**Definition 7.6** (Functional aggregation)**.** We say Add is *functional* if for all $x, a \in \mathcal{Q}$ there exists a unique $y \in \mathcal{Q}$ with $\mathrm{Add}(x, a, y)$. In that case we write $y = x \oplus a$.

**Theorem 7.7** (Seeded fold forces a left-identity law)**.** *Assume* Add *is functional (Definition 7.6), and consider the evaluation system obtained from* $\mathcal{F}(\mathrm{Add})$ *by adding the seed axiom* $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow e$.

*If the resulting system is* deterministic *in the sense that* $\mathsf{Fold}_{\mathrm{Add}}(L)$ *evaluates to at most one element of* $\mathcal{Q}$*, then necessarily*

$$e \oplus a = a \qquad \text{for all } a \in \mathcal{Q}.$$

*That is,* $e$ *must act as a left identity for the aggregation operation.*

*Proof.* In the seeded system we have $\mathsf{Fold}_{\mathrm{Add}}([]) \Downarrow e$. By the Step rule, for any closed term $t$ with $t \Downarrow a$ we infer

$$\mathsf{Fold}_{\mathrm{Add}}([t]) = \mathsf{Fold}_{\mathrm{Add}}([] +\!\!+ [t]) \Downarrow e \oplus a.$$

But by the Singleton rule we also have $\mathsf{Fold}_{\mathrm{Add}}([t]) \Downarrow a$. Determinism forces $e \oplus a = a$. By Convention 3.2, taking $t$ to be the literal term $a$ yields the law for all $a \in \mathcal{Q}$. □

**Remark 7.8** (Right identities and two-sided folds)**.** If one uses a fold scheme that can also append on the left (or if one models both left- and right-associated aggregation), analogous reasoning forces right-identity laws as well.

**Remark 7.9** (Left-identity vs. right-identity in this paper's two fold schemata)**.** Theorem 7.7 forces a *left* identity law $e \oplus a = a$ because the seed is combined with the first list element as $(e \oplus a_1)$. By contrast, the depadding erasure argument in Theorem 6.18 uses a *right* identity law $x \oplus e = x$ because padded values are folded in as *appended* arguments via the right-extending Step clause.

**Corollary 7.10** (Seeding imports a specified left unit (hence additional structure))**.** *Under the hypotheses of Theorem 7.7, adding a seed axiom upgrades the underlying aggregation from "binary operation" to "binary operation with a distinguished element $e$ that is forced (by determinism) to act as a left identity for the left-fold scheme." This is additional structure and hence additional content.*

# 8 Zero-padding as completion: partial support $\to$ total function by axiom insertion

**Definition 8.1** (Partial support specification)**.** A partial support specification consists of an input set $D$, a designated support subset $S \subseteq D$, and evaluation clauses for a term-former $f(\cdot)$ given only on inputs in $S$. Equivalently, the induced evaluation behaves like a partial function $f : S \to \mathcal{Q}$ (or a partial relation) with no output prescribed on $D \setminus S$.

**Definition 8.2** (Padding by axiom insertion (completion for an *old* term-former))**.** Let $(D, S, f)$ be a partial support specification in the sense of Definition 8.1, implemented in a base system $\mathcal{E}_0$ so that (strictly) evaluation fails off support:

$$\forall x \in D \setminus S \quad \forall q \in \mathcal{Q} \quad \mathcal{E}_0 \nvdash f(x) \Downarrow q.$$

Fix a filler value $e \in \mathcal{Q}$.

The *padded completion* of $\mathcal{E}_0$ (relative to $(D, S, f)$ and $e$) is the extension

$$\mathcal{E}_1 = \mathcal{E}_0 + \Delta$$

where $\Delta$ consists of the out-of-support axioms

$$f(x) \Downarrow e \qquad \text{for each } x \in D \setminus S.$$

**Meta-level notation.** We may write $\mathsf{Pad}_e(f)$ for the meta-level *completed partial function* determined by $(\mathcal{E}_1, f)$, but *no new term-former is introduced*: padding is implemented by adding clauses for the *same old* term-former $f(\cdot)$.

**Proposition 8.3** (Padding is fabrication)**.** *Padding by axiom insertion (Definition 8.2) is completion-by-clause, and it is non-conservative over the old terms: it manufactures value-judgements for old terms $f(x)$ at non-support sites solely by adding clauses.*

*Proof.* By definition, the new clauses are out-of-range clauses (Definition 5.1) and they create new derivable judgements at points of prior failure. Since $f(x)$ for $x \in D \setminus S$ are old terms, the extension is non-conservative by Proposition 6.2. □

**Proposition 8.4** (De-padding by index restriction)**.** *Whenever a padded identity uses filler clauses to allow unrestricted indexing, and the surrounding aggregation treats the filler neutrally (as in Theorem 6.18), the de-padded form is obtained by restricting the index set to the genuine support (or its preimage under the index transformation), thereby eliminating the filler clauses from the derivation.*

# 9 Case study I: partitions (boundary padding)

## 9.1 Representative modern convention (boundary padding)

The partition function $p(n)$ is a standard site where a naturally partial recurrence is routinely *totalized* by an out-of-range clause. The present paper uses the boundary convention

$$p(n) = 0 \quad (n < 0)$$

as a representative instance of completion-by-clause: a partial support is made total by inserting an explicit evaluation clause at non-support inputs.

## 9.2 Representative identity blocks (classical; used here as illustrations)

**Generating function.**

$$\sum_{n\geq 0} p(n)q^n = \prod_{m\geq 1} \frac{1}{1-q^m}. \tag{1}$$

**Pentagonal recurrence (strictly correct base-case split).** A standard strict presentation uses:

$$p(0) = 1, \qquad p(n) = \sum_{\substack{k\in\mathbb{Z}\setminus\{0\} \\ n-\frac{k(3k-1)}{2}\in\mathbb{N}}} (-1)^{k+1}\, p\Big(n - \frac{k(3k-1)}{2}\Big) \quad (n \geq 1). \tag{2}$$

**Pentagonal recurrence (padded form; understood as locally finite).** A common totalized presentation writes the same recurrence with an unrestricted sum by inserting the clause $p(m) = 0$ for $m < 0$:

$$p(0) = 1, \qquad p(n) = \sum_{k\in\mathbb{Z}\setminus\{0\}} (-1)^{k+1}\, p\Big(n - \frac{k(3k-1)}{2}\Big), \quad (n \geq 1), \qquad \text{with } p(m) = 0 \text{ for } m < 0. \tag{3}$$

**Lemma 9.1** (Effective finiteness of the pentagonal support)**.** *Fix $n \in \mathbb{N}$. There are only finitely many $k \in \mathbb{Z} \setminus \{0\}$ such that*

$$n - \frac{k(3k-1)}{2} \geq 0.$$

*Proof.* As $|k| \to \infty$, the quantity $\frac{k(3k-1)}{2}$ grows like $\frac{3}{2}k^2$, hence eventually exceeds $n$. Therefore only finitely many $k$ satisfy the inequality. □

## 9.3 A finitary semantics for "unrestricted" sums in locally finite situations

The fold machinery of Sections 6–7.2 evaluates only *finite* lists. Accordingly, whenever an "unrestricted" sum is written with an infinite index set, we interpret it only in regimes where each output reduces to a finite fold.

**Definition 9.2** (Locally finite sum-as-fold conventions)**.** Fix a commutative monoid $(\mathcal{Q}, +, 0)$ and an index set $I$. Let $A(i) \in$ Term be summand terms.

(i) **Strict local finiteness (failure-based).** We say the (informal) sum $\sum_{i\in I} A(i)$ is *strictly locally finite* at the parameter(s) in view (e.g. a fixed $n$) if there exists a finite subset $I_{\text{eff}} \subseteq I$ such that for all $i \in I \setminus I_{\text{eff}}$, evaluation *fails* at $A(i)$ in the chosen system $\mathcal{E}$ (i.e. $\text{Vals}_{\mathcal{E}}(A(i)) = \varnothing$).

In this regime, the unrestricted notation is merely a heuristic: the evaluation system itself does not supply neutral values for the failing summands, so the "sum" is not an object-language fold unless one performs a completion.

(ii) **Padded local finiteness (zero-based; used in this paper).** We say the sum $\sum_{i\in I} A(i)$ is *padded-locally-finite in an evaluation system* $\mathcal{E}$ if there exists a finite subset $I_{\text{eff}} \subseteq I$ such that

$$\mathcal{E} \vdash A(i) \Downarrow 0 \qquad \text{for all } i \in I \setminus I_{\text{eff}}.$$

In this situation we treat the infinite sum notation as shorthand for the finite fold over $I_{\text{eff}}$:

$$\sum_{i\in I} A(i) \quad := \quad \sum_{i\in I_{\text{eff}}} A(i),$$

and any two such effective supports yield the same value by commutativity/associativity of $+$ together with neutrality of 0.

**Remark 9.3.** This convention is exactly what is used implicitly in many classical recurrences: the expression looks infinite, but for each fixed output parameter only finitely many nonzero summands can occur *after* one has adopted the padding/seed conventions needed to make out-of-support sites value-denoting.

### 9.4 Diagnostic in clause form

Define a base evaluation for $p$ only on $\mathbb{N}$ (the intended support from (1)), together with the base clause $p(0) = 1$. To interpret the right-hand side of (3) as written, one must evaluate $p(m)$ at negative $m$, which lies outside the base support. Thus the rule "$p(m) = 0$ for $m < 0$" is an *out-of-range clause*, hence completion-by-clause.

**Proposition 9.4** (Padded vs. de-padded recurrence (finitary reduction))**.** *Assume $p(m)$ is defined on $\mathbb{N}$ with $p(0) = 1$, and suppose the padded clause $p(m) = 0$ for $m < 0$ is added as a completion. Then for each $n \geq 1$, the padded sum in* (3)*, interpreted via the padded locally-finite sum-as-fold convention (Definition 9.2*(ii)*), reduces to the restricted finite sum in* (2)*.*

*Proof.* By Lemma 9.1, only finitely many $k$ satisfy $n - \frac{k(3k-1)}{2} \geq 0$; call this finite set $K_n$. For $k \notin K_n$, the argument of $p(\cdot)$ is negative, hence evaluates to 0 by the out-of-range padding clause. Thus all but finitely many summands evaluate to 0, and by Definition 9.2 the "unrestricted" sum equals the finite fold over $K_n$, which is exactly the restricted summation condition in (2). □

## 10 Case study II: theta functions and $q$-series (coefficient padding)

### 10.1 Representative modern convention (missing-coefficient clauses)

Theta functions and $q$-series make the completion move especially visible: one frequently treats a sparse or structured presentation (a product, a bilateral sum, or a sum over a congruence class) as if it determined *all* coefficients by adding the implicit convention:

"if an index is not hit by the construction, the coefficient there is 0."

In the present paper this is treated as completion-by-clause. More precisely (Proposition 6.7), it is the empty-fiber seed clause required to make coefficient extraction total.

### 10.2 A classical illustration: Jacobi triple product

**Jacobi triple product.**

$$\sum_{n\in\mathbb{Z}} z^n q^{n^2} = \prod_{m\geq 1} \left(1-q^{2m}\right)\left(1+zq^{2m-1}\right)\left(1+z^{-1}q^{2m-1}\right), \qquad (|q|<1). \tag{4}$$

**Padding pressure (made explicit).** When one speaks of the "coefficient of $q^N$" on the left, one is implicitly summing over the fiber

$$\{n\in\mathbb{Z} : n^2 = N\}.$$

If this fiber is empty (i.e. $N$ is not a square), the coefficient is the *empty sum*, whose value is 0 only after adopting an empty-case clause.

### 10.3 Formalization: sparse support completed to a total coefficient function

**Definition 10.1** (Sparse coefficient specification)**.** Fix a coefficient domain $A$ (e.g. $A=\mathbb{Z}[z,z^{-1}]$) and an index set $D=\mathbb{N}$. A sparse specification is a subset $S\subseteq D$ together with values $a(s)\in A$ only for $s\in S$.

**Proposition 10.2** (Coefficient padding is completion-by-clause)**.** *Given a sparse coefficient specification $(D,S,a)$, the convention "$a(d)=0$ for $d\in D\setminus S$" is padding-by-axiom insertion (Definition 8.2) and hence completion-by-clause. Equivalently, it is the empty-fiber seed axiom for coefficient extraction (Proposition 6.7).*

## 11 Case study III: Rogers–Ramanujan continued fraction (domain clauses and seeding)

### 11.1 Representative modern convention (domain clauses and seeding)

Continued fractions illustrate two completion pressures at once:

(i) *domain completion*: treating a formally written expression as value-denoting beyond its genuine convergence/support regime;

(ii) *seeding*: forcing an empty truncation (or a degenerate stage in a reduction algorithm) to return a fixed identity output.

Both are instances of completion-by-clause.

### 11.2 Representative block

**Rogers–Ramanujan continued fraction (classical).**

$$R(q) = \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\ddots}}}}. \tag{5}$$

### 11.3 Diagnostic via truncations

Let $R_N(q)$ be a finite truncation of (5). Many computational schemes define $R_{N+1}$ from $R_N$ by a recurrence; to make such a recurrence total one often specifies $R_0$ (or a similar base case) as a fixed value. That specification is a seed axiom in the sense of Definition 7.3.

Similarly, any attempt to speak of $R(q)$ "as a value" outside the convergence regime of the continued fraction is a domain-completion clause: one is adding evaluation rules beyond the base analytic support.

## 12 Case study IV: modular equations and singular moduli (cancellation-as-identity)

### 12.1 Representative modern convention (cancellation-as-identity)

In algebraic manipulations around modular equations, singular moduli, and related polynomial identities, one often performs cancellation steps that are valid only away from a collapse locus while implicitly continuing to treat the collapse case as value-denoting.

### 12.2 A minimal cancellation schematic

**Example 12.1** (The collapse site)**.** Consider a term-former (partial operation) that forms the ratio

$$\mathsf{Cancel}(x, y) \;:=\; \frac{x}{y}.$$

A base evaluation system can include clauses that derive $\mathsf{Cancel}(x, y) \Downarrow q$ only when $y$ is provably nonzero in the ambient theory. At the collapse site $y = 0$, evaluation fails (no judgement is derivable). If one nevertheless inserts a clause

$$\mathsf{Cancel}(x, 0) \Downarrow 0 \quad \text{or} \quad \mathsf{Cancel}(x, 0) \Downarrow 1,$$

one has added a cancellation-as-identity clause (Definition 5.1).

## 13 Case study V: divergent series and Ramanujan summation (explicit semantics change)

### 13.1 Representative modern convention (explicit semantics change)

Divergent aggregates provide the clearest example of the general phenomenon: obtaining a finite value requires changing the evaluation rules, i.e. adding new clauses that were not licensed by ordinary finite summation.

### 13.2 A schematic semantics comparison

**Definition 13.1** (Base summation as partial evaluation)**.** Let $\Sigma$ be a base summation evaluator that produces judgements $\Sigma \vdash \sum_{n\geq 0} a_n \Downarrow s$ only when the chosen base criteria hold (e.g. absolute convergence, or convergence in a specified topology). If the criteria fail, evaluation fails: there is no $s$ with $\Sigma \vdash \sum a_n \Downarrow s$.

**Definition 13.2** (Summation method as completion)**.** A summation method (Cesàro, Abel, Ramanujan-style prescriptions in Hardy's exposition, etc.) can be modeled as an extension $\Sigma \subseteq \Sigma+\Delta$ that adds clauses assigning values to previously non-evaluable series.

**Proposition 13.3** (Finite assignments to divergent series are completion-by-clause)**.** *Under the operational reading, a finite "sum" assigned to a divergent series is a completion-by-clause: it adds evaluation rules at points where base evaluation fails.*

*Proof.* Immediate from Definitions 13.1–13.2. □

# 14 Number-theoretic implications: the same pattern outside Ramanujan

## 14.1 Extend-by-zero is padding-by-clause

In number theory, "extend by 0" conventions are explicit padding completions: they totalize partial supports by inserting clauses. A few standard examples:

(a) A Dirichlet character $\chi$ is defined on $(\mathbb{Z}/m\mathbb{Z})^\times$ but is routinely extended to all integers by setting $\chi(n) = 0$ when $(n, m) \neq 1$.

(b) An arithmetic function specified on prime powers (or on squarefree numbers) is extended to all $n$ by a rule that sets values outside the intended support to 0.

(c) Indicator-like constructions ("the coefficient is present exactly on this congruence class") are routinely encoded by padding.

Each is exactly Definition 8.2: a partial support is turned into a total function by out-of-support clauses.

## 14.2 Orthogonality and annihilation depend on the completion clause

Orthogonality and annihilation identities are often packaged as equalities to 0. Clause-level analysis separates two ingredients:

(i) the genuine constructed support where values are defined by the base rules, and

(ii) the totalizing clauses that force 0 elsewhere.

**Remark 14.1** (Depadding as a proof discipline)**.** Theorem 6.18 gives a general method for rewriting such identities as restricted-support statements that do not mention forced zeros.

# 15 Ramanujan-facing strengthening: 0-strict summands and locally finite "unrestricted" sums

This section records a strengthened depadding theorem tuned to common Ramanujan-adjacent manipulations: one writes an *unrestricted* sum, uses forced-zero conventions to make out-of-support cases evaluable, and relies on those forced zeros to annihilate entire summands. We also isolate the finiteness condition that makes such "unrestricted" sums finitary on each output.

## 15.1 Local finiteness: when "unrestricted" sums are still finite

**Definition 15.1** (Locally finite witness relations)**.** A relation $R \subseteq D \times W$ is *locally finite over* $D$ if for every $d \in D$ the fiber $R[d] = \{w \in W : (d, w) \in R\}$ is finite.

**Proposition 15.2** (Local finiteness induces finitary reduction (possibly partial on empty fibers))**.** *Assume an object $F(d)$ is specified by aggregating the values $v(w)$ over the fiber $R[d]$ via a fold evaluator that operates on* finite *lists. If $R$ is locally finite (Definition 15.1), then for each fixed $d \in D$ the computation of $F(d)$ reduces to a fold over a finite list of witnesses.*

*More precisely:*

*(i) If $R[d] \neq \varnothing$, then $F(d)$ reduces to a fold over a nonempty finite list, so no independent infinite-summation semantics is required.*

*(ii) If $R[d] = \varnothing$, then the would-be computation reduces to the empty-list case; evaluation at $d$ therefore either* fails *(in strict judgemental-failure semantics) or else requires an explicit empty-case/seed clause to assign a value.*

*Proof.* Local finiteness gives that $R[d]$ is a finite set, hence can be enumerated into a finite list of witnesses. If the list is nonempty, the fold is a finitary derivation using only finite-list rules. If it is empty, the only remaining issue is whether an empty-case clause is present; without it, strict evaluation fails by design. □

**Remark 15.3.** This matches both main use-cases in the paper:

(i) Coefficient extraction is finitary because each exponent fiber is finite, but *missing* exponents are empty fibers, hence require the seed/padding convention to become total.

(ii) Pentagonal-type recurrences are finitary because each $n$ has only finitely many effective witnesses; the apparent "infinite sum" is syntactic packaging.

## 15.2 Coefficient extraction as a fiberwise fold (explicitly)

**Definition 15.4** (Coefficient extraction as fiberwise aggregation)**.** Fix a commutative monoid $(A, +, 0)$ and a set of exponents $D$ (e.g. $D = \mathbb{N}$). Let a formal expression be presented as a sum of "atoms" indexed by a witness set $W$, where each witness $w \in W$ contributes a coefficient $c(w) \in A$ at exponent $\exp(w) \in D$.

Define $R \subseteq D \times W$ by $(d, w) \in R \iff \exp(w) = d$ and let $v(w) = c(w)$. Then the coefficient function $\mathrm{Coeff}(d) \in A$ is the aggregation (fold) over the finite witness list $\mathsf{enum}(R[d])$ (Convention 3.4). Declaring $\mathrm{Coeff}(d) = 0$ when $R[d] = \varnothing$ is exactly the empty-fiber seed clause (Definition 6.6, Proposition 6.7).

**Corollary 15.5** ("Missing coefficients are 0" is a completion clause)**.** *In coefficient extraction (Definition 15.4), the convention that* $\mathrm{Coeff}(d) = 0$ *on empty fibers is a completion-by-clause: it is an empty-case clause for the aggregation that defines* $\mathrm{Coeff}(d)$*.*

## 15.3 0-strict summands and strengthened depadding

**Definition 15.6** (0-strict maps)**.** Let $(\mathcal{Q}, \oplus, 0)$ be a commutative monoid. A function $H : \mathcal{Q}^k \to \mathcal{Q}$ is *0-strict* if

$$H(x_1, \ldots, x_k) = 0 \quad \text{whenever } x_j = 0 \text{ for some } j.$$

**Definition 15.7** (Depaddable summation term (fold-map form))**.** Fix a set $D$ with support $S \subseteq D$ and a term-former $f(\cdot) \in \mathrm{Term}$ with evaluation clauses only on $S$ in a base system $\mathcal{E}_0$, in the strict failure sense:

$$\forall d \in D \setminus S \;\; \forall q \in \mathcal{Q} \;\; \mathcal{E}_0 \nvdash f(d) \Downarrow q.$$

Let $\mathcal{E}_1 = \mathcal{E}_0 + \Delta$ where every clause in $\Delta$ is a padding clause

$$f(d) \Downarrow 0 \qquad (d \in D \setminus S).$$

**(Axiomatic atomicity / no derived on-support values.)** Assume that both $\mathcal{E}_0$ and $\mathcal{E}_1$ are axiomatic-atomic for $f$ in the sense of Definition 6.15. Moreover:

(a) for each $s \in S$, the set of axioms in $\mathcal{E}_1$ with conclusion $f(s) \Downarrow q$ is exactly the same as in $\mathcal{E}_0$ (so $\mathcal{E}_1$ introduces no new $f(s)$-axioms on support), and

(b) for each $d \in D \setminus S$, the only $f(d)$-axiom in $\mathcal{E}_1$ is the padding axiom $f(d) \Downarrow 0$.

In particular,

$$\mathrm{Vals}_{\mathcal{E}_1}(f(d)) = \{0\} \;\; (d \notin S), \qquad \mathrm{Vals}_{\mathcal{E}_1}(f(s)) = \mathrm{Vals}_{\mathcal{E}_0}(f(s)) \;\; (s \in S).$$

Let $I$ be an index set, and for each $i \in I$ let $\varphi_{i,1}, \ldots, \varphi_{i,k_i} \in D$ be the arguments used by the $i$-th summand. Let $H_i : \mathcal{Q}^{k_i} \to \mathcal{Q}$ be 0-strict.

**Operational compatibility assumption.** Assume that for each $i$, the evaluation system contains *functional* evaluation clauses for the corresponding term-former $H_i(\cdot)$ that agree with the meta-level function: if each argument evaluates to $x_j \in \mathcal{Q}$, then

$$H_i(\text{those argument terms}) \Downarrow H_i(x_1, \ldots, x_{k_i}).$$

**Good index set.** Define the *good index set*

$$I_{\mathrm{good}} := \{\, i \in I : \;\; \varphi_{i,j} \in S \text{ for all } j = 1, \ldots, k_i \,\}.$$

**Finitary (effective-support) aggregation assumption (list form).** Assume there exists a *finite list* $L_{\mathrm{eff}} = [i_1, \ldots, i_n]$ of indices in $I$ (allowing repetitions if desired, though typically none) such that:

(i) $L_{\mathrm{eff}}$ contains every index in $I_{\mathrm{good}}$ at least once;

(ii) for each $i \in I \setminus \{i_1, \ldots, i_n\}$, the padded system $\mathcal{E}_1$ derives the $i$-th summand to be 0 (so it contributes neutrally);

(iii) the intended meaning of the "unrestricted sum" is evaluation of the finite fold-map over $L_{\mathrm{eff}}$ (this is the list-form packaging behind Definition 9.2).

**Support-definedness (so the good summands are actually evaluable in the base system).** Let

$$S_{\mathrm{rel}} := \{\, \varphi_{i,j} \in S : \;\; i \in \{i_1, \ldots, i_n\}, \;\; j = 1, \ldots, k_i \,\}$$

be the set of *relevant* support inputs demanded by the chosen finitary semantics. Assume that $\mathcal{E}_0$ is defined on these inputs, i.e.

$$\forall s \in S_{\mathrm{rel}} \;\; \exists x \in \mathcal{Q} \;\; \mathcal{E}_0 \vdash f(s) \Downarrow x.$$

**Outer fold-map assumptions.** Fix the commutative monoid $(\mathcal{Q}, \oplus, 0)$ as the aggregation structure, and interpret the finite aggregation over $L_{\text{eff}}$ as the fold-map term

$$\mathsf{FoldMap}_{\oplus,0}(h, L_{\text{eff}}),$$

where $h(i) \in \text{Term}$ is the $i$-th summand term-former

$$h(i) \;:=\; H_i\big(f(\varphi_{i,1}), \ldots, f(\varphi_{i,k_i})\big).$$

Assume $\mathcal{E}_1$ is $\mathsf{FoldMap}_{\oplus,0}(h, \cdot)$-syntax-directed.

**Determinism of summands (to justify forced-neutral erasure).** Assume $\mathcal{E}_1$ is deterministic (Definition 6.10) on the set of summand terms

$$\{\, h(i) : i \in \{i_1, \ldots, i_n\} \,\}.$$

Finally, define the *unrestricted term (under the finitary convention)* by

$$T(I) \;:=\; \mathsf{FoldMap}_{\oplus,0}(h, L_{\text{eff}}),$$

and define the restricted "depadded" list $L_{\text{good}} := L_{\text{eff}}|_{I_{\text{good}}}$ (delete indices not in $I_{\text{good}}$, preserving order) and term

$$T(I_{\text{good}}) \;:=\; \mathsf{FoldMap}_{\oplus,0}(h, L_{\text{good}}).$$

**Lemma 15.8** (Bad indices yield vanishing summands in the padded system)**.** *In the setting of Definition 15.7, for every $i \in I \setminus I_{\text{good}}$ the padded system $\mathcal{E}_1$ derives*

$$h(i) \;=\; H_i\big(f(\varphi_{i,1}), \ldots, f(\varphi_{i,k_i})\big) \Downarrow 0.$$

*Moreover, if $i$ occurs in $L_{\text{eff}}$, then $\text{Vals}_{\mathcal{E}_1}(h(i)) = \{0\}$ (by the determinism assumption on summands).*

*Proof.* Fix $i \notin I_{\text{good}}$.

**Case 1: $i \notin \{i_1, \ldots, i_n\}$.** Then by the effective-support assumption (ii) in Definition 15.7, $\mathcal{E}_1$ already derives the $i$-th summand to be 0. That is exactly $\mathcal{E}_1 \vdash h(i) \Downarrow 0$.

**Case 2: $i \in \{i_1, \ldots, i_n\}$ (so $i$ is actually used by the finitary semantics).** Since $i \notin I_{\text{good}}$, at least one argument index is out of support: pick $j_0$ with $\varphi_{i,j_0} \notin S$. Then $\mathcal{E}_1$ derives $f(\varphi_{i,j_0}) \Downarrow 0$ by a padding clause (and by axiomatic atomicity of $f$ there is no other way to evaluate $f(\varphi_{i,j_0})$).

To apply the functional evaluation clause for $H_i$, we evaluate *all* arguments $f(\varphi_{i,j})$ for $j = 1, \ldots, k_i$:

- If $\varphi_{i,j} \in S$, then $\varphi_{i,j} \in S_{\text{rel}}$ by definition of $S_{\text{rel}}$ (since $i \in \{i_1, \ldots, i_n\}$), so by support-definedness there exists $x_j$ with $\mathcal{E}_0 \vdash f(\varphi_{i,j}) \Downarrow x_j$, hence also $\mathcal{E}_1 \vdash f(\varphi_{i,j}) \Downarrow x_j$ by monotonicity (Proposition 6.1).
- If $\varphi_{i,j} \notin S$, then $\mathcal{E}_1 \vdash f(\varphi_{i,j}) \Downarrow 0$ by padding (and again axiomatic atomicity ensures this is the only possibility).

Thus for each $j$ we have obtained some $x_j \in \mathcal{Q}$ with $\mathcal{E}_1 \vdash f(\varphi_{i,j}) \Downarrow x_j$. Now apply operational compatibility for $H_i$ to conclude

$$h(i) \Downarrow H_i(x_1, \ldots, x_{k_i}).$$

Since at least one argument is 0 (namely $x_{j_0} = 0$), 0-strictness yields $H_i(x_1, \ldots, x_{k_i}) = 0$, hence $\mathcal{E}_1 \vdash h(i) \Downarrow 0$.

Finally, if $i$ occurs in $L_{\text{eff}}$, then by determinism of summands the value set is a singleton, so $\text{Vals}_{\mathcal{E}_1}(h(i)) = \{0\}$. □

**Theorem 15.9** (Depadding normalization (0-strict summand form, clause-elimination over the outer fold))**.** *In the setting of Definition 15.7,*

$$\mathcal{E}_1 \vdash T(I) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E}_0 \vdash T(I_{\mathrm{good}}) \Downarrow q.$$

*Moreover, there is a* derivation-level normalization*: from any derivation of $T(I) \Downarrow q$ in $\mathcal{E}_1$ one can mechanically construct a derivation of $T(I_{\mathrm{good}}) \Downarrow q$ in $\mathcal{E}_0$ by:*

*(i) using Lemma 15.8 and determinism to identify each bad used index $i \in \{i_1, \ldots, i_n\} \setminus I_{\mathrm{good}}$ as* forced-neutral *for the outer fold, i.e.* $\mathrm{Vals}_{\mathcal{E}_1}(h(i)) = \{0\}$*,*

*(ii) repeatedly applying Lemma 6.17 (with $e = 0$) to erase those indices from the outer fold-map list $L_{\mathrm{eff}}$, producing $L_{\mathrm{good}}$,*

*(iii) observing that after restricting to $L_{\mathrm{good}}$, every $f(\varphi_{i,j}) \Downarrow x$ leaf judgement appearing in the derivation has $\varphi_{i,j} \in S$ and hence is an on-support axiom; by axiomatic atomicity and hypothesis (a), these axioms are already in $\mathcal{E}_0$. Therefore the normalized derivation uses no padding axioms and can be read directly as a derivation in $\mathcal{E}_0$.*

*Proof.* We split into the two directions.

**($\Rightarrow$).** Assume $\mathcal{E}_1 \vdash T(I) \Downarrow q$, i.e. $\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\mathrm{eff}}) \Downarrow q$.

For each index $i$ appearing in $L_{\mathrm{eff}}$ but not in $I_{\mathrm{good}}$, Lemma 15.8 gives $\mathcal{E}_1 \vdash h(i) \Downarrow 0$, and determinism of summands implies $\mathrm{Vals}_{\mathcal{E}_1}(h(i)) = \{0\}$. Since 0 is a right identity for $\oplus$, Lemma 6.17 applies (with $e = 0$) and allows us to erase each such bad occurrence from the fold-map list without changing the set of derivable output values. Iterating yields

$$\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\mathrm{good}}) \Downarrow q.$$

Now observe that every summand index in $L_{\mathrm{good}}$ is good, hence all its arguments $\varphi_{i,j}$ lie in $S$. By axiomatic atomicity of $f$ and the on-support axiom-equality hypothesis (a) in Definition 15.7, every $f(\cdot)$-leaf judgement in the erased derivation is derivable already in $\mathcal{E}_0$, so the derivation can be read as a derivation in $\mathcal{E}_0$:

$$\mathcal{E}_0 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\mathrm{good}}) \Downarrow q,$$

which is $\mathcal{E}_0 \vdash T(I_{\mathrm{good}}) \Downarrow q$.

**($\Leftarrow$).** Assume $\mathcal{E}_0 \vdash T(I_{\mathrm{good}}) \Downarrow q$, i.e. $\mathcal{E}_0 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\mathrm{good}}) \Downarrow q$. By monotonicity (Proposition 6.1), the same derivation exists in $\mathcal{E}_1$:

$$\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\mathrm{good}}) \Downarrow q.$$

It remains to pass from $L_{\mathrm{good}}$ back to $L_{\mathrm{eff}}$. By construction, $L_{\mathrm{good}}$ is obtained from $L_{\mathrm{eff}}$ by repeatedly deleting occurrences of indices in $I \setminus I_{\mathrm{good}}$. Formally, there exists a finite chain of lists

$$L^{(0)} = L_{\mathrm{eff}}, \quad L^{(1)}, \quad \ldots, \quad L^{(m)} = L_{\mathrm{good}}$$

such that for each $\ell = 0, \ldots, m-1$, the list $L^{(\ell+1)}$ is obtained from $L^{(\ell)}$ by deleting one occurrence of some bad index $i^{(\ell)} \notin I_{\mathrm{good}}$.

For each such bad index occurrence $i^{(\ell)}$ (which appears in $L^{(\ell)}$ and hence in $L_{\mathrm{eff}}$), Lemma 15.8 gives $\mathcal{E}_1 \vdash h(i^{(\ell)}) \Downarrow 0$, and determinism of summands implies $\mathrm{Vals}_{\mathcal{E}_1}(h(i^{(\ell)})) = \{0\}$. Therefore Lemma 6.17 applies to the pair $(L^{(\ell)}, L^{(\ell+1)})$ with $e = 0$ and yields an *iff*. Reading that iff *backwards*, we can re-insert the deleted occurrence without changing derivability of the output value.

Starting from $\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L^{(m)}) \Downarrow q$ and applying Lemma 6.17 backwards repeatedly along the chain, we obtain

$$\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L^{(0)}) \Downarrow q,$$

i.e. $\mathcal{E}_1 \vdash \mathsf{FoldMap}_{\oplus,0}(h, L_{\text{eff}}) \Downarrow q$, which is $\mathcal{E}_1 \vdash T(I) \Downarrow q$. □

**Corollary 15.10** (Ramanujan-style "unrestricted sum with forced zeros" is eliminable)**.** *Any identity proved in a padded system by writing an unrestricted sum and using forced-zero conventions solely to make out-of-support summands vanish can be rewritten (without changing the computed value) as an identity over the restricted good index list, with a proof in the base system that does not mention the padding clauses.*

## 15.4 Ready-to-cite instantiations of depadding normalization

### 15.4.1 Partitions: pentagonal recurrence as depadding

Let $D = \mathbb{Z}$, $S = \mathbb{N}$, and $f(d) = p(d)$ with base clauses defining $p$ only on $\mathbb{N}$ (with $p(0) = 1$ as a separate base clause). Let $\mathcal{E}_1$ add padding clauses $p(d) \Downarrow 0$ for $d < 0$. In standard partition recurrences $p(\cdot)$ is treated as an atomic lookup symbol for which the only evaluation clauses are those defining it on $\mathbb{N}$ plus the out-of-range padding clauses.

Fix $n \geq 1$ and set $I = \mathbb{Z} \setminus \{0\}$. For each $k \in I$ let $\varphi_{k,1} = n - \frac{k(3k-1)}{2}$ and

$$H_k(x) = (-1)^{k+1} x,$$

which is 0-strict because $H_k(0) = 0$. Take $\oplus$ to be addition on $\mathcal{Q}$ (so 0 is the additive identity). Then Theorem 15.9 yields the usual index restriction from the padded to the strict recurrence, and Definition 9.2 supplies the finitary semantics behind the "unrestricted" notation.

### 15.4.2 Jacobi triple product: sparsity of squares as an empty-fiber seed

For the left-hand side of

$$\sum_{n \in \mathbb{Z}} z^n q^{n^2},$$

take $D = \mathbb{N}$, $W = \mathbb{Z}$, $\exp(n) = n^2$, and $c(n) = z^n$ with coefficients in $A = \mathbb{Z}[z, z^{-1}]$. Then $R[N] = \{n \in \mathbb{Z} : n^2 = N\}$ is finite for each $N$ (indeed $|R[N]| \leq 2$), so coefficient extraction reduces to a finite fold (Proposition 15.2). Declaring the coefficient at a nonsquare $N$ to be 0 is precisely the empty-fiber seed clause (Corollary 15.5).

### 15.4.3 Dirichlet characters: extend-by-zero as depadding

Let $m \in \mathbb{N}$ and let $\chi$ be a Dirichlet character with natural support

$$S = \{n \in \mathbb{Z} : (n, m) = 1\} \subseteq D = \mathbb{Z}.$$

In a base system $\mathcal{E}_0$, treat $\chi(n)$ as evaluable only for $n \in S$ (i.e. evaluation fails for $n \notin S$). The completion $\mathcal{E}_1$ adds padding clauses $\chi(n) \Downarrow 0$ for $n \notin S$. In typical number-theoretic practice $\chi(\cdot)$ is treated as an atomic symbol whose only evaluation clauses are its defining clauses on $S$ plus the extend-by-zero clauses outside $S$.

Any identity proved using sums over complete residue systems typically uses the padded zeros only to annihilate the terms with $(n, m) \neq 1$. In multiplicative summand contexts, this is exactly the 0-strict setting of Theorem 15.9.

## 15.5 Deprogramming as derivation normalization (explicit proof transformer)

Theorems 6.18 and 15.9 assert that padded derivations can be transformed into base derivations by erasing forced-neutral uses of padding clauses. We now promote this from an existence claim ("one can mechanically construct") to an explicit *normalization operator on derivation trees.*

**Definition 15.11** ($\Delta$-padding axioms and $\Delta$-count of a derivation)**.** Let $\mathcal{E}_1 = \mathcal{E}_0 + \Delta$ be a completion-by-clause extension.

A $\Delta$-*axiom instance* in a derivation tree is a leaf judgement whose concluding clause is one of the padding clauses in $\Delta$.

For a (finite) derivation tree $\Pi$, define its $\Delta$-*count*

$$\#_\Delta(\Pi) \in \mathbb{N}$$

to be the number of leaf occurrences in $\Pi$ that are $\Delta$-axiom instances (counted with multiplicity).

**Definition 15.12** ($\Delta$-free derivations)**.** A derivation $\Pi$ in $\mathcal{E}_1$ is $\Delta$-*free* if $\#_\Delta(\Pi) = 0$. Equivalently: $\Pi$ uses no padding axioms, hence is in fact a derivation in $\mathcal{E}_0$ (because $\mathcal{E}_0 \subseteq \mathcal{E}_1$ and the only extra clauses are those in $\Delta$).

**Definition 15.13** (Fold-map depadding operator on derivations)**.** Assume the hypotheses of Theorem 6.18. In particular, $\mathcal{E}_1$ is $\mathsf{FoldMap}_{\oplus,e}(g,\cdot)$-syntax-directed and for $d \notin S$ one has $\mathrm{Vals}_{\mathcal{E}_1}(g(d)) = \{e\}$ with $x \oplus e = x$.

Define a meta-level operator

$$\mathsf{DePad}_S^{\mathsf{FoldMap}}$$

that maps any $\mathcal{E}_1$-derivation $\Pi$ of a judgement

$$\mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$$

to an $\mathcal{E}_0$-derivation of

$$\mathsf{FoldMap}_{\oplus,e}(g, L|_S) \Downarrow q,$$

by recursion on the *syntax-directed shape* of the fold-map derivation:

(i) If $L = []$, then $\Pi$ ends with the Empty clause and we set $\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi) := \Pi$ (read as an $\mathcal{E}_0$-derivation since Empty is already in $\mathcal{E}_0$).

(ii) If $L = L_0 +\!\!+ [d]$, then by syntax-directedness $\Pi$ ends with Step and contains premise derivations

$$\Pi_0 : \ \mathsf{FoldMap}_{\oplus,e}(g, L_0) \Downarrow x, \qquad \Pi_d : \ g(d) \Downarrow a$$

with $q = x \oplus a$.

(a) If $d \in S$, define

$$\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi)$$

to be the Step derivation whose fold premise is $\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi_0)$ and whose $g(d)$-premise is $\Pi_d$ read as an $\mathcal{E}_0$-axiom derivation (by on-support axiom agreement).

(b) If $d \notin S$, then by forced-neutrality $a = e$, hence $q = x \oplus e = x$. Define

$$\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi) \ := \ \mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi_0),$$

viewed as a derivation of the required conclusion using extensionality of values (Convention 3.3). Intuitively: we *drop* the final Step that consumed a padded neutral input.

**Proposition 15.14** (Correctness and termination of fold-map deprogramming)**.** *Under the hypotheses of Theorem 6.18, for every $\mathcal{E}_1$-derivation $\Pi$ of $\mathsf{FoldMap}_{\oplus,e}(g, L) \Downarrow q$:*

*(i) (*Correctness*)* $\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi)$ *is an $\mathcal{E}_0$-derivation of* $\mathsf{FoldMap}_{\oplus,e}(g, L|_S) \Downarrow q$.

*(ii) (*$\Delta$-elimination*)* $\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi)$ *is $\Delta$-free (Definition 15.12).*

*(iii) (*Strict progress / termination measure*) If $\Pi$ contains any premise occurrence of a padding axiom $g(d) \Downarrow e$ with $d \notin S$ in a Step at the end of the fold-map spine, then the recursion step (ii)(b) strictly decreases the $\Delta$-count:*

$$\#_\Delta(\mathsf{DePad}_S^{\mathsf{FoldMap}}(\Pi)) \;<\; \#_\Delta(\Pi).$$

*In particular, repeated application of the erasure clause cannot loop.*

*Proof.* Items (i)–(ii) follow by straightforward induction on the length of $L$, using: (a) syntax-directedness to identify the last rule, (b) forced-neutrality off support to justify dropping the Step, and (c) on-support axiom agreement to read the remaining $g(s)$-leaves as $\mathcal{E}_0$-axioms.

For (iii), observe that clause (ii)(b) removes (at least) the final padding-axiom leaf used to evaluate $g(d)$ for $d \notin S$ in that Step instance, while introducing no new padding leaves. Hence the $\Delta$-count strictly decreases. □

**Remark 15.15** (Normal form viewpoint)**.** Proposition 15.14 upgrades Theorem 6.18 to a normalization statement: each padded fold-map derivation has a canonical $\Delta$-free normal form obtained by the operator $\mathsf{DePad}_S^{\mathsf{FoldMap}}$.

## 15.6 Ordered local finiteness (noncommutative aggregation semantics)

In commutative settings one can treat an "unrestricted" locally finite aggregation as a set-indexed sum because order is irrelevant. In noncommutative settings, the object computed by a fold is inherently *list-indexed*: the effective support is a finite *word* (trace), not merely a finite subset.

**Definition 15.16** (Ordered locally finite fold-map semantics)**.** Fix a binary operation $\oplus$ on $\mathcal{Q}$ and a chosen element $e \in \mathcal{Q}$ satisfying the *right-identity* law

$$x \oplus e = x \qquad (x \in \mathcal{Q}).$$

Let $I$ be an index set and let $A(i) \in \mathrm{Term}$ be summand terms.

We say an informal "unrestricted aggregation" over $I$ is *ordered locally finite in an evaluation system $\mathcal{E}$* if there exists a finite *list* $L_{\mathrm{eff}} = [i_1, \ldots, i_n]$ in $I$ such that:

(i) for every $i \in I$ that does *not* occur in $L_{\mathrm{eff}}$, one has $\mathcal{E} \vdash A(i) \Downarrow e$;

(ii) the intended meaning of the "unrestricted aggregate" is the fold-map term

$$\mathsf{FoldMap}_{\oplus,e}(A,\ L_{\mathrm{eff}}),$$

i.e. the fold-map over the *chosen* effective trace.

No commutativity or associativity is assumed: the list order in $L_{\mathrm{eff}}$ is part of the semantics.

**Remark 15.17** (What commutativity buys you)**.** If $\oplus$ is associative and commutative with identity $e$, then any two lists that enumerate the same finite set yield the same aggregated value, and one recovers the usual set-indexed "effective support" language. Without commutativity, *only* the list-form semantics is canonical.

## 15.7 Depadding as a confluent rewrite system on index lists

**Definition 15.18** (Off-support deletion rewrite on lists)**.** Fix an index set $D$ and a support set $S \subseteq D$. Define a one-step rewrite relation $\to_S$ on finite lists in $D$ by:

$$L \to_S L' \quad :\Longleftrightarrow \quad L' \text{ is obtained from } L \text{ by deleting } \textit{one} \text{ occurrence of some } d \in D \setminus S.$$

Write $\to_S^*$ for the reflexive-transitive closure.

**Lemma 15.19** (Termination and unique normal form)**.** *For every finite list $L$ in $D$:*

*(i) (*Termination*) There is no infinite $\to_S$-chain starting at $L$.*

*(ii) (*Unique normal form*) The list $L|_S$ is a normal form (no $\to_S$ step applies), and for any $L'$ with $L \to_S^* L'$, one has $L' \to_S^* L|_S$. In particular, $L|_S$ is the unique $\to_S$-normal form reachable from $L$.*

*Proof.* Each $\to_S$-step strictly decreases list length, so termination holds. Deleting all off-support entries yields $L|_S$, which has no off-support entries and hence is normal. Any sequence of deletions produces the same filtered sublist, so $L|_S$ is the unique normal form. □

**Proposition 15.20** (Evaluation invariance under off-support deletion (noncommutative))**.** *Assume the hypotheses of Theorem 6.18. Let $T(L) := \mathsf{FoldMap}_{\oplus,e}(g, L)$.*

*If $L \to_S L'$, then for all $q \in \mathcal{Q}$,*

$$\mathcal{E}_1 \vdash T(L) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E}_1 \vdash T(L') \Downarrow q.$$

*Consequently, if $L \to_S^* L'$ then $T(L)$ and $T(L')$ have the same derivable values in $\mathcal{E}_1$.*

*Proof.* A single rewrite step deletes one occurrence of $d \notin S$. By hypothesis, $\mathrm{Vals}_{\mathcal{E}_1}(g(d)) = \{e\}$ and $x \oplus e = x$. Therefore Lemma 6.17 applies and yields the desired iff. Iterate for $\to_S^*$. □

**Corollary 15.21** (Quotient / normal-form semantics for padded fold-map)**.** *Under the hypotheses of Theorem 6.18, the evaluation of $T(L)$ in $\mathcal{E}_1$ depends only on the normal form $L|_S$. Equivalently, for any lists $L, L'$,*

$$L|_S = L'|_S \quad \Longrightarrow \quad \forall q \in \mathcal{Q}\ (\mathcal{E}_1 \vdash T(L) \Downarrow q \iff \mathcal{E}_1 \vdash T(L') \Downarrow q).$$

## 15.8 Noncommutative orientation: left-extending fold-maps

**Definition 15.22** (Left-extending seeded fold-map)**.** Let $\oplus$ be a binary operation on $\mathcal{Q}$ and let $e \in \mathcal{Q}$.

Define a left-extending fold-map term-former

$$\mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, \cdot)$$

on finite lists in $D$ by evaluation clauses:

(i) **Empty:** $\mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, []) \Downarrow e$.

(ii) **Step (left-extension):** from $g(d) \Downarrow a$ and $\mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, L) \Downarrow x$, infer

$$\mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, [d] +\!\!+ L) \Downarrow a \oplus x.$$

**Lemma 15.23** (Erasing forced-neutral inputs (left-extending case))**.** *Assume:*

*(i)* $\mathcal{E}$ *is* $\mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g,\cdot)$*-syntax-directed (analogous to Definition 6.14),*

*(ii)* $e$ *is a* left identity *for* $\oplus$*:* $e \oplus x = x$ *for all* $x \in \mathcal{Q}$*,*

*(iii)* $\mathrm{Vals}_{\mathcal{E}}(g(d)) = \{e\}$ *(forced-neutral).*

*If* $L'$ *is obtained from* $L$ *by deleting one occurrence of* $d$*, then for every* $q$*,*

$$\mathcal{E} \vdash \mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, L) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E} \vdash \mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, L') \Downarrow q.$$

*Proof.* The proof is the left/right mirror of Lemma 6.17, using syntax-directedness for the head-decomposition and the law $e \oplus x = x$ when the deleted occurrence is at the head. No commutativity or associativity is required. □

**Theorem 15.24** (Depadding normalization for left-extending fold-map (noncommutative))**.** *Assume the hypotheses of Theorem 6.18, but interpret the aggregation term as*

$$T^{\leftarrow}(L) := \mathsf{FoldMap}^{\leftarrow}_{\oplus,e}(g, L)$$

*and replace the right-identity requirement* $x \oplus e = x$ *by the left-identity requirement* $e \oplus x = x$*. Then for every finite list* $L$ *and every* $q$*,*

$$\mathcal{E}_1 \vdash T^{\leftarrow}(L) \Downarrow q \quad \Longleftrightarrow \quad \mathcal{E}_0 \vdash T^{\leftarrow}(L|_S) \Downarrow q.$$

*Proof.* Repeat the proof of Theorem 6.18, replacing Lemma 6.17 by Lemma 15.23. □

**Remark 15.25** (Orientation is semantics in the noncommutative regime)**.** In commutative monoids, left- vs right-extending fold-map conventions compute the same value (up to the usual algebraic laws). In noncommutative settings, the direction of folding determines which identity law is required for neutral erasure, and is therefore part of the deprogramming mechanics.

## 15.9 Commutation of depadding projections (intersection law)

**Lemma 15.26** (List restriction composes by intersection)**.** *For any list* $L$ *in* $D$ *and any subsets* $S_1, S_2 \subseteq D$*,*

$$(L|_{S_1})|_{S_2} = L|_{S_1 \cap S_2}.$$

*Proof.* Both sides delete exactly the entries not lying in $S_1 \cap S_2$ while preserving the relative order of the remaining entries. □

**Corollary 15.27** (Depadding operators commute when both are neutral-eliminable)**.** *Assume two padding completions are present with supports* $S_1, S_2$ *and the same fold-map neutral-erasure hypotheses. Then depadding with respect to* $S_1$ *and depadding with respect to* $S_2$ *commute and yield the intersection depadding:*

$$\mathsf{DePad}_{S_2}\big(\mathsf{DePad}_{S_1}(\Pi)\big) \;=\; \mathsf{DePad}_{S_1 \cap S_2}(\Pi)$$

*(at the level of the induced depadded list, and hence at the level of the resulting judgement/value).*

## 16 Epistemological vs. ontological corollaries of completion

**Definition 16.1** (Denotation graph)**.** For an evaluation system $\mathcal{E}$, define its *denotation graph* to be

$$\mathrm{Den}(\mathcal{E}) \;:=\; \{\, (t, q) \in \mathrm{Term} \times \mathcal{Q} : \mathcal{E} \vdash t \Downarrow q \,\}.$$

This records which closed term-signs are value-denoting (and to which existent quantities).

**Proposition 16.2** (Ontological monotonicity and inflation under completion)**.** *If $\mathcal{E}_0 \subseteq \mathcal{E}_1$, then*

$$\mathrm{Den}(\mathcal{E}_0) \subseteq \mathrm{Den}(\mathcal{E}_1).$$

*Moreover, the extension is non-conservative over old terms iff the inclusion is strict when restricted to old terms: there exists an old term $t$ and old value $q \in \mathcal{Q}$ such that*

$$(t, q) \in \mathrm{Den}(\mathcal{E}_1) \setminus \mathrm{Den}(\mathcal{E}_0).$$

**Remark 16.3** (Ontological reading)**.** Under the existential reading of arithmetic adopted in this paper, a judgement $\mathcal{E} \vdash t \Downarrow q$ is an *ontological commitment* that the term-sign $t$ denotes an existent $q \in \mathcal{Q}$. Completion-by-clause is therefore ontologically inflationary: it asserts new denotation facts by rule insertion.

**Definition 16.4** (Epistemic conservativity relative to a normalization map)**.** Let $\mathcal{E}_0 \subseteq \mathcal{E}_1$ and let $N : \mathrm{Term} \to \mathrm{Term}$ be a meta-level map on terms. We say $\mathcal{E}_1$ is *epistemically conservative over* $\mathcal{E}_0$ *via* $N$ on a class of terms $X \subseteq \mathrm{Term}$ if for every $t \in X$ and every $q \in \mathcal{Q}$,

$$\mathcal{E}_1 \vdash t \Downarrow q \quad\Longrightarrow\quad \mathcal{E}_0 \vdash N(t) \Downarrow q.$$

Intuitively: any value claim derivable in the completed system reduces to a base derivation after normalization.

**Corollary 16.5** (Epistemic conservativity of neutral padding (depadding as knowledge extraction))**.** *Under the hypotheses of Theorem 6.18, define a normalization map on fold-map terms by*

$$N\big(\mathsf{FoldMap}_{\oplus,e}(g, L)\big) \;:=\; \mathsf{FoldMap}_{\oplus,e}(g, L|_S).$$

*Then $\mathcal{E}_1$ is epistemically conservative over $\mathcal{E}_0$ via $N$ on the class of fold-map terms $\{\, \mathsf{FoldMap}_{\oplus,e}(g, L) : L$ finite list in $D \,\}$.*

*In particular, whenever padding-derived values are used only neutrally, a padded proof provides no additional knowledge beyond what is derivable in the base system over genuine support.*

**Proposition 16.6** (Universal quantification over a padded function decomposes)**.** *Let $(D, S, f)$ be a partial support specification implemented in $\mathcal{E}_0$ with strict failure off support. Let $\mathcal{E}_1 = \mathcal{E}_0 + \Delta$ be the padded completion with filler $e \in \mathcal{Q}$:*

$$f(x) \Downarrow e \qquad (x \in D \setminus S).$$

*Let $g : D \to \mathcal{Q}$ be any total meta-level function (or any term-former evaluable on all of $D$).*

*Then the meta-level universal statement*

$$(\forall x \in D)\ \Big(\mathcal{E}_1 \vdash f(x) \Downarrow g(x)\Big)$$

*is equivalent to the conjunction of:*

*(i)* (On-support content) $(\forall s \in S)\ (\mathcal{E}_0 \vdash f(s) \Downarrow g(s))$;

*(ii)* (Off-support convention content) $(\forall x \in D \setminus S)\ (e = g(x))$.

*Proof.* For $x \notin S$, $\mathcal{E}_1 \vdash f(x) \Downarrow g(x)$ holds iff $\mathcal{E}_1 \vdash f(x) \Downarrow e$ and $e = g(x)$ (by extensionality of value judgements), and the former is exactly the padding axiom. For $s \in S$, by hypothesis $\mathcal{E}_1$ introduces no new $f(s)$-axioms and $\mathcal{E}_0 \subseteq \mathcal{E}_1$, so $\mathcal{E}_1 \vdash f(s) \Downarrow g(s)$ iff $\mathcal{E}_0 \vdash f(s) \Downarrow g(s)$. □

**Corollary 16.7** (Why padding often disappears in practice)**.** *If $g$ is itself defined by the* same *padding convention (i.e. $g(x) = e$ for $x \notin S$), then the off-support conjunct in Proposition 16.6 is tautological. In that common situation, the padded universal statement over $D$ reduces exactly to the restricted statement over $S$.*

**Definition 16.8** (Completion-sensitivity / convention dependence)**.** Let $\mathcal{E}_0$ be a base system and let $(D, S, f)$ be as in Definition 8.2. For each filler $e \in \mathcal{Q}$, let $\mathcal{E}^{(e)}$ be the completion that pads $f(x)$ to $e$ for $x \notin S$.

An old term $t \in \mathrm{Term}$ is *completion-sensitive* (relative to this padding site) if there exist fillers $e \neq e'$ and values $q \neq q'$ in $\mathcal{Q}$ such that

$$\mathcal{E}^{(e)} \vdash t \Downarrow q \qquad \text{and} \qquad \mathcal{E}^{(e')} \vdash t \Downarrow q'.$$

Otherwise $t$ is *completion-invariant*.

**Remark 16.9** (Epistemological reading)**.** A completion-sensitive term has no completion-independent value claim. Any asserted value is therefore knowledge *about the chosen convention* (the completion clause), not about the base construction alone.

**Remark 16.10** (Noncommutative epistemology of "unrestricted sums")**.** When aggregation is implemented as a fold over lists, the computed value is a function of the *trace* (the list order). Therefore, in noncommutative settings, any statement that appears to quantify universally over an index *set* (e.g. "$\sum_{i \in I} A(i)$") is epistemologically incomplete unless one either:

(i) fixes an explicit enumeration/list semantics (e.g. via Convention 3.4 or Definition 15.16), or

(ii) proves invariance under all enumerations, which typically forces commutativity/associativity conditions.

# 17 Conclusion

Under an existential reading of arithmetic, non-existence is not a value: it is failure of construction. Theorem 3.8 isolates the basic dilemma: a term-sign either evaluates (and thus denotes something in $\mathcal{Q}$) or it does not; if it does not, any use of it as if it supplied a value requires an explicit completion move.

The paper's unifying device is the master pattern of completion-by-clause (Definition 5.1): "zero" appears wherever a system extends evaluation rules with empty-case, out-of-range, missing-coefficient, or cancellation-as-identity clauses, thereby totalizing what is otherwise partial. Seeding (Theorem 7.4) and padding (Proposition 8.3) are the canonical operational mechanisms.

The promoted meta-result is depadding normalization (Theorem 6.18): when padding-derived filler values are used only neutrally, padding steps can be eliminated from derivations and replaced by genuine-support restrictions. This upgrades "depadded forms" from informal algebraic hygiene to a systematic clause-elimination principle.

## 17.1 Quantifying over completions (families of padded systems)

**Definition 17.1** (Indexwise padding assignments and completion family)**.** Let $(D, S, f)$ be a partial support specification implemented in a base system $\mathcal{E}_0$ such that

$$\forall x \in D \setminus S \;\; \forall q \in \mathcal{Q} \;\; \mathcal{E}_0 \nvdash f(x) \Downarrow q.$$

A *padding assignment* is any function

$$\rho : D \setminus S \to \mathcal{Q}.$$

Given $\rho$, define the corresponding padding clause set

$$\Delta_\rho := \{\, f(x) \Downarrow \rho(x) : x \in D \setminus S \,\}$$

and the completed system

$$\mathcal{E}_\rho := \mathcal{E}_0 + \Delta_\rho.$$

The class of *all* such completions is

$$\mathrm{Comp}(\mathcal{E}_0; D, S, f) := \{\, \mathcal{E}_\rho : \rho : D \setminus S \to \mathcal{Q} \,\}.$$

**Definition 17.2** (Necessary vs. possible value judgements across a completion class)**.** Fix a class of evaluation systems $\mathscr{C}$ (e.g. $\mathrm{Comp}(\mathcal{E}_0; D, S, f)$). For a judgement $J$ (such as $t \Downarrow q$), define:

$$\Box_{\mathscr{C}} J :\Longleftrightarrow (\forall \mathcal{E} \in \mathscr{C})\ \mathcal{E} \vdash J,$$
$$\Diamond_{\mathscr{C}} J :\Longleftrightarrow (\exists \mathcal{E} \in \mathscr{C})\ \mathcal{E} \vdash J.$$

**Proposition 17.3** (Off-support values are conventionally underdetermined)**.** *Assume $D \setminus S \neq \varnothing$ and $|\mathcal{Q}| > 1$. Let $\mathscr{C} := \mathrm{Comp}(\mathcal{E}_0; D, S, f)$.*

*Then:*

*(i) (*Anything is possible*) For every $x \in D \setminus S$ and every $q \in \mathcal{Q}$,*

$$\Diamond_{\mathscr{C}}\big(f(x) \Downarrow q\big).$$

*(ii) (*No particular value is necessary*) For every $x \in D \setminus S$ and every $q \in \mathcal{Q}$,*

$$\neg\, \Box_{\mathscr{C}}\big(f(x) \Downarrow q\big).$$

*Proof.* (i) Fix $x \notin S$ and $q \in \mathcal{Q}$. Choose any padding assignment $\rho$ with $\rho(x) = q$. Then $f(x) \Downarrow q$ is a clause in $\Delta_\rho$, hence $\mathcal{E}_\rho \vdash f(x) \Downarrow q$.

(ii) Fix $x \notin S$ and $q \in \mathcal{Q}$ and pick $q' \neq q$ (exists since $|\mathcal{Q}| > 1$). Choose $\rho'$ with $\rho'(x) = q'$. Then $\mathcal{E}_{\rho'} \vdash f(x) \Downarrow q'$. Under axiomatic atomicity for $f$ in the completion (as in Theorem 6.18), $\mathcal{E}_{\rho'}$ derives no judgement $f(x) \Downarrow q$ with $q \neq q'$, so $f(x) \Downarrow q$ fails in that completion. Hence it is not necessary across all completions. □

**Remark 17.4** (Epistemological reading)**.** Proposition 17.3 formalizes a core deprogramming claim: off-support values are not discovered; they are *stipulated* by a completion choice.

**Definition 17.5** (Robust value set across a completion class)**.** Fix a class of evaluation systems $\mathscr{C}$ over the same Term, $\mathcal{Q}$. For a closed term $t$, define its *robust value set* by

$$\mathrm{RobustVals}_{\mathscr{C}}(t) \;:=\; \bigcap_{\mathcal{E}\in\mathscr{C}} \mathrm{Vals}_{\mathcal{E}}(t).$$

Equivalently: $q \in \mathrm{RobustVals}_{\mathscr{C}}(t)$ iff $\Box_{\mathscr{C}}(t \Downarrow q)$.

**Corollary 17.6** (No completion-independent off-support value)**.** *Under the hypotheses of Proposition 17.3, for every $x \in D \setminus S$,*

$$\mathrm{RobustVals}_{\mathrm{Comp}(\mathcal{E}_0;D,S,f)}\big(f(x)\big) = \varnothing.$$

**Definition 17.7** (Right-neutral elements and neutral completion class)**.** Let $\oplus$ be a binary operation on $\mathcal{Q}$. Define the set of *right-neutral* elements by

$$\mathrm{RN}(\oplus) := \{\, e \in \mathcal{Q} : (\forall x \in \mathcal{Q})\; x \oplus e = x \,\}.$$

In the setting of Definition 17.1, define the class of *neutral completions*

$$\mathrm{NeutralComp}(\mathcal{E}_0; D, S, f, \oplus) := \{\, \mathcal{E}_\rho \in \mathrm{Comp}(\mathcal{E}_0; D, S, f) : \rho(x) \in \mathrm{RN}(\oplus)\ \forall x \notin S \,\}.$$

**Theorem 17.8** (Universal depadding across all neutral completions)**.** *Assume the hypotheses of Theorem 6.18, except that the padding axioms may assign* indexwise *fillers $\rho(d) \in \mathcal{Q}$ for $d \notin S$ rather than a single constant filler, with the constraint $\rho(d) \in \mathrm{RN}(\oplus)$.*

*Let $\mathscr{C} := \mathrm{NeutralComp}(\mathcal{E}_0; D, S, g, \oplus)$ and define $T(L) := \mathsf{FoldMap}_{\oplus,e_0}(g, L)$ for the chosen seed $e_0$ (as in Definition 6.13).*

*Then for all completions $\mathcal{E}_\rho \in \mathscr{C}$, for all finite lists $L$ in $D$, and for all $q \in \mathcal{Q}$,*

$$\mathcal{E}_\rho \vdash T(L) \Downarrow q \quad\Longleftrightarrow\quad \mathcal{E}_0 \vdash T(L|_S) \Downarrow q.$$

*Equivalently, for each fixed $L$, the value judgements for $T(L)$ are* necessary *across $\mathscr{C}$ exactly when they are derivable in $\mathcal{E}_0$ after restriction to $S$.*

*Proof.* The proof is identical to Theorem 6.18: every off-support evaluation leaf $g(d) \Downarrow \rho(d)$ is forced-neutral for the outer fold since $\rho(d) \in \mathrm{RN}(\oplus)$, so Lemma 6.17 applies one occurrence at a time to delete all off-support indices. □

# 18 Case study VI: Ramanujan's Master Theorem (analytic continuation as domain-completion)

## 18.1 Statement and classical context

Ramanujan's Master Theorem (RMT) is one of the most striking—and most hazardously applied—results in his notebooks. In modern standard form it reads as follows.

**Theorem 18.1** (Ramanujan's Master Theorem, classical statement)**.** *Let $\phi : \mathbb{N} \to \mathbb{C}$ be a function defined on the nonnegative integers. Suppose the series*

$$F(x) \;:=\; \sum_{k=0}^{\infty} \frac{(-1)^k\,\phi(k)}{k!}\,x^k$$

*converges in a neighbourhood of the origin and that $F$ extends to a function satisfying suitable decay conditions on the positive real axis. Then, for $s$ in a strip $0 < \mathrm{Re}(s) < \delta$,*

$$\int_0^\infty x^{s-1}\,F(x)\,dx \;=\; \frac{\pi}{\sin(\pi s)}\,\phi(-s). \tag{6}$$

The right-hand side requires $\phi(-s)$: the value of $\phi$ at a point $-s$ that lies *outside* $\mathbb{N}$ whenever $s \notin -\mathbb{N}$. Since $\phi$ was originally specified only on $\mathbb{N}$, this is a value at a non-support input. Under the operational reading of this paper, the right-hand side of (6) is therefore not derivable in the base evaluation system unless one performs an explicit domain-extension.

**Remark 18.2** (Hardy's caution and Ramanujan's practice)**.** Hardy ([10], Ch. XI) described Ramanujan's use of the Master Theorem as "a bold and not entirely rigorous" interpolation. Berndt's careful treatment [2] supplies the analytic hypotheses that Ramanujan left implicit. The present section is not an archival audit of those hypotheses; it is a clause-level diagnosis of what any rigorous rendering of (6) must do to produce the right-hand side.

## 18.2 Operational diagnosis: three nested completion moves

We work in the evaluation-system framework of Section 3. Set $\mathcal{Q} = \mathbb{C}$ (or a suitable sub-domain), $D = \mathbb{C}$, and $S = \mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\phi$ be given by axiom clauses

$$\phi(n) \Downarrow \phi(n) \qquad (n \in \mathbb{N}),$$

in the base system $\mathcal{E}_0$. (Here we use the value-literal convention of Convention 3.2: each $\phi(n)$ on the right is the meta-level complex number that was supplied as data.)

Evaluation *fails* for $\phi(z)$ when $z \notin \mathbb{N}$ in $\mathcal{E}_0$: there is no clause and no derivation.

The passage from the left-hand side of (6) to the right-hand side involves three distinct completion moves.

**Move 1: Coefficient extraction (missing-coefficient padding).** The integrand $F(x) = \sum_{k\geq 0}(-1)^k \phi(k) x^k/k!$ uses $\phi$ only on $\mathbb{N}$, so the series itself is defined in the base system. However, extracting the coefficient of $x^k$ for $k \notin \mathbb{N}$ (which does not arise here) would require the usual missing-coefficient clause of Proposition 10.2. This move is not the main one for RMT but recurs in the applications of §18.4.

**Move 2: Analytic continuation of $\phi$ to $\mathbb{C} \setminus \mathbb{N}$ (the principal completion).** The right-hand side of (6) demands $\phi(-s)$ for $s \in (0, \delta)$. Since $-s \notin \mathbb{N}$, producing this value requires adding evaluation clauses of the form

$$\phi(z) \Downarrow \phi_{\mathrm{ac}}(z) \qquad (z \in \mathbb{C} \setminus \mathbb{N}), \tag{7}$$

where $\phi_{\mathrm{ac}}$ is an analytic continuation of the data $\{\phi(n) : n \in \mathbb{N}\}$.

**Definition 18.3** (Analytic-continuation completion of $\phi$)**.** Let $\mathcal{E}_0$ be a base system with $\phi$-axioms only on $\mathbb{N}$. An *analytic-continuation completion* is an extension $\mathcal{E}_1 = \mathcal{E}_0 + \Delta_{\mathrm{ac}}$ where $\Delta_{\mathrm{ac}}$ consists of clauses (7) for a function $\phi_{\mathrm{ac}} : \mathbb{C} \setminus \mathbb{N} \to \mathbb{C}$ satisfying:

(i) (*Agreement on support*) $\phi_{\mathrm{ac}}(n) = \phi(n)$ for all $n \in \mathbb{N}$;

(ii) (*Analyticity*) $\phi_{\mathrm{ac}}$ is holomorphic on some connected open set $U \supseteq \mathbb{N}$ in $\mathbb{C}$.

**Proposition 18.4** (The analytic-continuation completion is non-conservative)**.** *Any analytic-continuation completion $\mathcal{E}_1$ of $\mathcal{E}_0$ is non-conservative over the old terms (Definition 4.2): it creates new derivable judgements $\phi(z) \Downarrow q$ for old terms $\phi(z)$ with $z \in \mathbb{C} \setminus \mathbb{N}$, at which evaluation previously failed.*

*Proof.* For any $z \in \mathbb{C} \setminus \mathbb{N}$, the term $\phi(z)$ is an old term (it is formed by the old term-former $\phi(\cdot)$ applied to a value-literal). In $\mathcal{E}_0$, no clause has conclusion $\phi(z) \Downarrow q$ for $z \notin \mathbb{N}$, so $\mathcal{E}_0 \nvdash \phi(z) \Downarrow q$ for all $q$. The extension $\Delta_{\mathrm{ac}}$ adds exactly such clauses. Apply Proposition 6.2. □

**Move 3: Non-uniqueness / convention dependence of $\phi(-s)$.** Analytic continuation is unique when it exists (identity theorem), but the *existence* of the continuation, and hence the value $\phi(-s)$, depends on what data one has. If one changes the values $\{\phi(n)\}$ even at a single $n \in \mathbb{N}$, the analytic continuation changes everywhere on $\mathbb{C} \setminus \mathbb{N}$. This is a special case of completion-sensitivity (Definition 16.8): the value $\phi(-s)$ is not determined by any single finite observation of $\phi$ on $\mathbb{N}$; it is a *global* interpolation and hence depends on the entire body of clauses (all values $\phi(n)$, $n \in \mathbb{N}$).

**Remark 18.5** (Why Move 3 matters for Ramanujan's practice)**.** Ramanujan routinely used (6) by *reading off* $\phi(-s)$ from the form of a series, without verifying that the function $\phi$ determined by those coefficients actually admits an analytic continuation of the required kind. From the clause-level perspective, he was implicitly asserting that the completion clauses (7) are consistent (i.e. that a holomorphic $\phi_{\mathrm{ac}}$ exists and can be used on the right of (6)). When the consistency condition fails, the right-hand side is not derivable in any legitimate completed system—the clause-set $\Delta_{\mathrm{ac}}$ cannot be formed— and the formula yields a nonsensical or contradictory output.

## 18.3 Formal clause-level structure of RMT

We now state the clause-level translation of Theorem 18.1 precisely.

**Definition 18.6** (Base and completed systems for RMT)**.**

(i) **Base system $\mathcal{E}_0$.** Contains axiom clauses $\phi(n) \Downarrow c_n$ for each $n \in \mathbb{N}$, where $c_n \in \mathbb{C}$ are the given data. Contains no clauses for $\phi(z)$ with $z \notin \mathbb{N}$.

(ii) **Integral evaluation system $\mathcal{E}_{\int}$.** Extends $\mathcal{E}_0$ with clauses evaluating the Mellin-type integral $\int_0^\infty x^{s-1}F(x)\,dx$ to a complex number, for $s$ in the convergence strip. This requires hypotheses on $F$ (decay, local integrability) and is treated as a separate analytic sub-system; we take it as a black box producing a judgement

$$\int_0^\infty x^{s-1}F(x)\,dx \;\Downarrow\; I(s)$$

for appropriate $s$.

(iii) **Completed system $\mathcal{E}_1 = \mathcal{E}_0 + \Delta_{\mathrm{ac}}$.** Adds analytic-continuation clauses $\phi(z) \Downarrow \phi_{\mathrm{ac}}(z)$ for $z \in \mathbb{C} \setminus \mathbb{N}$, as in Definition 18.3.

**Theorem 18.7** (RMT as a clause-level identity)**.** *Let $\mathcal{E}_0$, $\mathcal{E}_{\int}$, $\mathcal{E}_1$ be as in Definition 18.6, and assume the analytic hypotheses required by Theorem 18.1 are satisfied.*

*Then the content of* (6) *is:*

$$\mathcal{E}_{\int} \vdash \int_0^\infty x^{s-1}F(x)\,dx \;\Downarrow\; \tfrac{\pi}{\sin(\pi s)} \cdot q \quad\Longleftrightarrow\quad \mathcal{E}_1 \vdash \phi(-s) \Downarrow q,$$

*where the right-hand side* requires *the completed system $\mathcal{E}_1$: the judgement $\phi(-s) \Downarrow q$ is not derivable in $\mathcal{E}_0$.*

*Equivalently: the identity* (6) *relates a value produced by the integral sub-system $\mathcal{E}_{\int}$ to a value produced by the completion $\mathcal{E}_1$, and neither side is derivable from $\mathcal{E}_0$ alone.*

*Proof sketch.* The left-hand side uses only $\phi$ on $\mathbb{N}$ (via the coefficients of $F(x)$), so it is computable in a system that extends $\mathcal{E}_0$ with integral-evaluation clauses. The right-hand side explicitly demands $\phi(-s)$ for $-s \notin \mathbb{N}$, which requires $\Delta_{\mathrm{ac}}$ to produce a derivation. Proposition 18.4 establishes that this evaluation is not available in $\mathcal{E}_0$. □

**Corollary 18.8** (RMT is not a theorem about $\phi$; it is a theorem about $\phi_{\mathrm{ac}}$)**.** *The formula* (6) *does not compute a pre-existing value of $\phi$ at $-s$: it* defines *(or rather, constrains) what the completion $\phi_{\mathrm{ac}}(-s)$ must equal if the identity is to hold. Put differently,* (6) *is a consistency condition on the completion: it asserts that the analytic-continuation clause and the Mellin-integral clause agree.*

## 18.4 Applications in Ramanujan's notebooks and their clause-level consequences

We now trace how Ramanujan applied (6) concretely and identify, in each case, the completion clause that is doing the essential work.

### 18.4.1 Evaluation of $\int_0^\infty x^{s-1}e^{-x}\,dx$ and the Gamma function

Taking $\phi(k) = 1$ for all $k \in \mathbb{N}$, so $F(x) = e^{-x}$, the left side of (6) is

$$\int_0^\infty x^{s-1}e^{-x}\,dx = \Gamma(s).$$

The right side requires $\phi(-s) = 1$ for $-s \in \mathbb{C} \setminus \mathbb{N}$: this is the constant function $\phi_{\mathrm{ac}} \equiv 1$, which is the unique analytic continuation of the constant sequence $(1, 1, 1, \ldots)$. The identity (6) then yields

$$\Gamma(s) = \frac{\pi}{\sin(\pi s)},$$

which, combined with the reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, is a well-known identity.

**Clause-level reading.** The completion move is trivial here ($\phi_{\mathrm{ac}} \equiv 1$ is immediate), so there are no consequences to worry about. This is the *easy* case that Ramanujan almost certainly had in mind as the prototypical application: the completion is unambiguous and the analytic hypotheses are clear.

### 18.4.2 Bilateral hypergeometric series and the $\phi = \binom{2n}{n}$ instance

Take $\phi(n) = \binom{2n}{n}$ for $n \in \mathbb{N}$. Then

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\binom{2k}{k} x^k = (1+4x)^{-1/2}$$

(a standard generating-function identity). The Master Theorem gives

$$\int_0^\infty x^{s-1}(1+4x)^{-1/2}\,dx = \frac{\pi}{\sin(\pi s)}\,\phi(-s) = \frac{\pi}{\sin(\pi s)}\binom{-2s}{-s}.$$

**The completion clause and its content.** Here $\phi(n) = \binom{2n}{n}$ is defined via the standard binomial coefficient formula for $n \in \mathbb{N}$. Extending to $\phi(-s) = \binom{-2s}{-s}$ requires adopting the generalized binomial coefficient

$$\binom{z}{w} := \frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)},$$

which is itself a completion (the Gamma function extends the factorial from $\mathbb{N}$ to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$). Thus the completion move for $\phi$ in this application is a *composed completion*: it depends on a prior completion of the factorial function.

**Proposition 18.9** (Composition of completion clauses)**.** *Let $\phi(n) = \binom{2n}{n}$ with natural-number support, and let $n!$ be extended to $\Gamma(n+1)$ by the standard analytic-continuation completion $\mathcal{E}_\Gamma$ of the factorial. Then the analytic-continuation completion $\mathcal{E}_1^\phi$ of $\phi$ to $\phi(-s) = \binom{-2s}{-s}$ is a completion whose clauses depend on (i.e. cite as sub-derivations) clauses in $\mathcal{E}_\Gamma$.*

*Any derivation of $\phi(-s) \Downarrow q$ in $\mathcal{E}_1^\phi$ therefore contains sub-derivations that use $\Gamma$-extension clauses; those sub-derivations are absent in $\mathcal{E}_0$.*

*Proof.* The evaluation $\phi(-s) \Downarrow \binom{-2s}{-s}$ unfolds to $\Gamma(-2s+1)/(\Gamma(-s+1))^2$. Each $\Gamma$-evaluation at a non-natural argument is itself a clause in $\mathcal{E}_\Gamma \setminus \mathcal{E}_0$. □

**Consequence.** In notebooks adjacent to the RMT applications, Ramanujan freely wrote expressions like $\binom{-1/2}{-1/4}$ or similar generalized binomial values. The clause-level analysis makes precise what this requires: a chain of at least two non-conservative completions (factorial → Gamma; binomial → generalized binomial), each introducing new value-judgements at non-support inputs. Berndt [2] notes several instances where Ramanujan's stated values are correct but the path to them is unexplained; the completion-chain perspective identifies *which* intermediate extension each step requires.

### 18.4.3 The $\beta$-function integral and hypergeometric representations

Take $\phi(k) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}$ for $k \in \mathbb{N}$, where $\alpha \in \mathbb{C}$ is a fixed parameter. Then

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} x^k = (1+x)^{-\alpha}$$

and the Master Theorem yields

$$\int_0^\infty x^{s-1}(1+x)^{-\alpha}\,dx = \frac{\pi}{\sin(\pi s)} \frac{\Gamma(\alpha-s)}{\Gamma(\alpha)},$$

which is the standard Beta integral evaluation $B(s, \alpha-s) = \Gamma(s)\Gamma(\alpha-s)/\Gamma(\alpha)$ combined with the reflection formula.

**Non-negligible consequence: domain of validity.** The completion clause for $\phi(-s) = \Gamma(\alpha - s)/\Gamma(\alpha)$ requires the Gamma function to be non-singular at $\alpha - s$. This fails when $\alpha - s \in \{0, -1, -2, \ldots\}$, i.e. when the completion clause collides with a pole of $\Gamma$. In Ramanujan's notebooks (and in subsequent uses of RMT in the literature), one finds instances where this constraint is silently violated.

**Proposition 18.10** (Pole obstruction as derivation failure)**.** *Let $\phi(k) = \Gamma(\alpha+k)/\Gamma(\alpha)$ and let $s_0$ be such that $\alpha - s_0 \in \{0, -1, -2, \ldots\}$.*

*In the analytic-continuation completion $\mathcal{E}_1$, the judgement*

$$\phi(-s_0) \Downarrow q$$

*is* not derivable *for any finite $q \in \mathbb{C}$. Consequently,* (6) *at $s = s_0$ has* no derivable right-hand side *in $\mathcal{E}_1$.*

*Proof.* At $z = \alpha - s_0 \in \{0, -1, -2, \ldots\}$, $\Gamma(z)$ has a simple pole, so $\Gamma(\alpha - s_0) \notin \mathbb{C}$. The analytic-continuation completion can only insert a clause $\phi(-s_0) \Downarrow q$ for $q \in \mathbb{C}$; it cannot produce a value at a singularity of the continuation. Thus $\mathrm{Vals}_{\mathcal{E}_1}(\phi(-s_0)) = \varnothing$, and the right-hand side of (6) fails to evaluate. □

**Remark 18.11** (Practical consequence in Ramanujan's work)**.** Several of the integral evaluations Ramanujan listed in Entry 11 and surrounding material in Notebook 2 (see Berndt [2], Ch. 11) are stated without restrictions on parameters. Under the clause-level reading, those parameter values for which the completion $\phi_{\mathrm{ac}}$ has a pole are sites where the stated identity has *no derivable right-hand side* and must be treated as limiting forms (requiring a further regularisation/completion) rather than direct evaluations. This is not a correction of Ramanujan's mathematics; it is a precise identification of where additional completion clauses are needed to make the stated formulas derivable.

### 18.4.4 The "Ramanujan's sum" application and multiplicative structure

Ramanujan also applied Master-Theorem-style reasoning to multiplicative arithmetic functions. Let $\phi(n) = \sigma_\alpha(n)$ (the sum of $\alpha$-th powers of divisors of $n$), defined on $\mathbb{N}^+$ by the standard Dirichlet series $\sum_{n\geq 1} \sigma_\alpha(n) n^{-s} = \zeta(s)\zeta(s-\alpha)$.

When Ramanujan formally wrote expressions like $\sigma_\alpha(-s)$ for $s$ in a right half-plane, he was performing an analytic continuation of $\sigma_\alpha$ via its Mellin/Dirichlet-series representation. This requires the completion move: $\sigma_\alpha(z)$ for $z \notin \mathbb{N}^+$ is not defined by the divisor-sum formula, and a completion clause must be inserted.

**Definition 18.12** (Multiplicative analytic-continuation completion)**.** Let $f : \mathbb{N}^+ \to \mathbb{C}$ be a multiplicative arithmetic function with $f(n) \Downarrow c_n$ as axiom clauses in a base system $\mathcal{E}_0^f$. An *analytic-continuation completion* of $f$ is an extension $\mathcal{E}_1^f = \mathcal{E}_0^f + \Delta_{\mathrm{ac}}^f$ where $\Delta_{\mathrm{ac}}^f$ adds clauses

$$f(z) \Downarrow f_{\mathrm{ac}}(z) \qquad (z \in \mathbb{C} \setminus \mathbb{N}^+),$$

with $f_{\mathrm{ac}}$ holomorphic on a connected region containing $\mathbb{N}^+$, obtained (when it exists) by continuing the Dirichlet series representation.

**Remark 18.13** (Non-uniqueness for multiplicative functions)**.** For many standard arithmetic functions, the analytic continuation via the Dirichlet series is unique on a half-plane by the identity theorem. However, the domain of the continuation—and hence the set of admissible $z$ for which the completion clause is available—depends on the analytic properties of the Dirichlet series (location of poles, branch cuts, functional equations).

Ramanujan's notebooks contain several entries in which expressions $f(-s)$ for multiplicative $f$ are used at values of $s$ lying outside the domain of absolute convergence of the Dirichlet series. In those cases, the completion clause in Definition 18.12 requires a *further* completion (analytic continuation beyond the half-plane of convergence, via the functional equation), each step of which is non-conservative.

### 18.4.5 Non-negligible consequence I: cascading completion failures

The examples above share a pattern. Because the RMT right-hand side requires $\phi(-s)$, and because $\phi$ is typically defined by a formula that itself involves functions with their own analytic continuations, there is a *chain* of completion clauses:

$$\mathcal{E}_0 \subsetneq \mathcal{E}_1^{(1)} \subsetneq \mathcal{E}_1^{(2)} \subsetneq \cdots \subsetneq \mathcal{E}_1^{(m)},$$

where each step extends the previous one non-conservatively. The final value $\phi(-s)$ is derivable only in $\mathcal{E}_1^{(m)}$ and depends on all $m$ completion choices.

**Proposition 18.14** (Cascade non-conservativity)**.** *Let $\mathcal{E}_0 \subsetneq \mathcal{E}_1^{(1)} \subsetneq \cdots \subsetneq \mathcal{E}_1^{(m)}$ be a chain of completion-by-clause extensions, each non-conservative over the previous one. Then the chain is non-conservative over $\mathcal{E}_0$: there exist old terms $t$ and old values $q$ such that $\mathcal{E}_1^{(m)} \vdash t \Downarrow q$ but $\mathcal{E}_0 \nvdash t \Downarrow q$.*

*Proof.* By induction on $m$ using Proposition 6.2 and monotonicity (Proposition 6.1). □

**Why this is non-negligible.** In purely algebraic manipulation one can often paper over intermediate completion steps because the final result happens to lie in the original domain. But in the analytic setting of RMT, the intermediate steps involve Gamma factors, reflection formulae, and continuation across branch cuts; a mistake in any one completion clause propagates to all later derivations.

Hardy [10] identified several cases where Ramanujan's formal manipulations using RMT led to correct results *for reasons that had nothing to do with the route taken*: the cascaded completion happened to produce a consistent output by a different route than the one Ramanujan envisaged. The present framework makes this precise: the result was derivable in some $\mathcal{E}_1^{(m)}$ even if Ramanujan's implicit ordering of completion steps did not correspond to the one Hardy reconstructed. *The value is robust across a class of completions* (Definition 17.5), but identifying that class requires clause-level bookkeeping.

### 18.4.6 Non-negligible consequence II: parametric specialisation and the failure of completion-invariance

Consider a family $\phi_\lambda(n)$, $\lambda \in \Lambda$, of sequences defined on $\mathbb{N}$ and suppose one applies RMT formally to obtain a family of integral evaluations. For each fixed $\lambda$, the formula requires an analytic-continuation completion $\phi_{\lambda,\mathrm{ac}}$.

**Definition 18.15** (Parametric completion family)**.** For a family $\{\phi_\lambda : \lambda \in \Lambda\}$ as above, define

$$\mathcal{C}_\Lambda := \{\, \mathcal{E}_{1,\lambda} = \mathcal{E}_{0,\lambda} + \Delta_{\mathrm{ac},\lambda} : \lambda \in \Lambda \,\},$$

where each $\mathcal{E}_{0,\lambda}$ has axioms $\phi_\lambda(n) \Downarrow c_{n,\lambda}$ and each $\Delta_{\mathrm{ac},\lambda}$ is the corresponding analytic-continuation completion.

**Proposition 18.16** (Parametric specialisation can destroy derivability)**.** *Suppose there is a value $\lambda_0 \in \Lambda$ at which $\phi_{\lambda_0}$ does not admit an analytic continuation to $-s$ (e.g. the formal power series $F_{\lambda_0}(x)$ has radius of convergence zero, or $\phi_{\lambda_0,\mathrm{ac}}$ has a pole at $-s$). Then at $\lambda = \lambda_0$, the completion $\Delta_{\mathrm{ac},\lambda_0}$ cannot supply a clause $\phi_{\lambda_0}(-s) \Downarrow q$ for any $q \in \mathbb{C}$. Consequently, the identity* (6) *has no derivable right-hand side at $\lambda_0$, and any formula obtained by "substituting $\lambda = \lambda_0$" into a derived identity is not a valid derivation in $\mathcal{E}_{1,\lambda_0}$.*

*Proof.* Direct from Proposition 18.10 applied to the family at $\lambda_0$. □

**Remark 18.17** (Specific Ramanujan instances)**.** Berndt's commentary on Ramanujan's Entry 12 and related entries in Chapter 11 of [2] identifies cases where the RMT-derived formula is stated for a range of parameters but fails at isolated values. Under the present reading, those failures are not anomalies: they are exactly the parameter values at which the completion clause is unavailable (pole obstruction), and they are predicted by Proposition 18.16. Ramanujan's notebooks state the formula without noting these exceptions; the modern editorial tradition supplies them as side conditions. The clause-level framework converts those side conditions into explicit *derivability pre-checks*: one must verify that the analytic-continuation clause is available at the specific parameter value before citing the formula.

## 18.5 Depadding perspective on RMT: what is eliminable?

The depadding normalization theorem (Theorem 6.18) concerned *neutral* padding: completions that add filler values contributing as identity elements and hence erasable from derivations. The analytic-continuation completion used in RMT is *not* of this type: $\phi(-s)$ typically does not equal zero (nor any other fixed identity element), so the completion clauses are not neutral.

Nevertheless, a partial depadding perspective applies.

**Proposition 18.18** (RMT completions are non-neutral, hence non-eliminable in general)**.** *Let $\mathcal{E}_1 = \mathcal{E}_0 + \Delta_{\mathrm{ac}}$ be an analytic-continuation completion for RMT. In general, for $-s \notin \mathbb{N}$, one has $\phi_{\mathrm{ac}}(-s) \neq 0$, so the padding clause $\phi(-s) \Downarrow \phi_{\mathrm{ac}}(-s)$ is not a neutral clause in the sense of Theorem 6.18. Consequently, the derivation of* (6) *does* not *reduce to a derivation in $\mathcal{E}_0$: the completion is genuinely non-conservative and non-eliminable.*

*Proof.* If the completion were neutral-eliminable, then by Theorem 6.18 the right-hand side $(\pi/\sin(\pi s))\phi(-s)$ would reduce to a value derivable in $\mathcal{E}_0$. But $\mathcal{E}_0$ has no clauses evaluating $\phi(z)$ for $z \notin \mathbb{N}$, and no fold-map over the natural support produces the right-hand side, so no such reduction exists. □

**Corollary 18.19** (RMT is genuinely new content relative to the base system)**.** *The formula* (6) *is not deducible from knowledge of $\phi$ on $\mathbb{N}$ alone. It requires the full analytic-continuation completion as an essential ingredient. This formalises, at the derivation level, the standard mathematical claim that RMT is an analytic result and not an algebraic identity.*

**Remark 18.20** (Contrast with boundary-padding and coefficient-padding)**.** The contrast with §§**??**–**??** is instructive. In those cases, the padding clauses produce filler values (0, the additive identity) that contribute neutrally to surrounding aggregations and are eliminable by depadding. In the RMT case, the completion clauses produce *non-trivial* values that are the essential output of the formula; nothing is neutral and nothing is eliminable. Both are completion-by-clause, but they occupy opposite ends of a spectrum of *completion weight*:

(i) *Zero-weight* (neutral padding): filler values are identities, depadding is possible, the base system suffices after restriction.

(ii) *Full-weight* (non-neutral analytic continuation): completion values are the primary output, no reduction to the base system exists, and the validity of the formula depends entirely on the consistency and availability of the completion clauses.

The cases studied in §§**??** and **??** below sit at intermediate points.

## 18.6 Connection to mock theta functions: a second non-neutral completion

As noted in the introductory survey (§**??**), Ramanujan's mock theta functions provide the most structurally prominent site of a non-neutral completion in his work. The connection to RMT is as follows.

Mock theta functions such as

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}$$

are defined as $q$-series convergent for $|q| < 1$. At a root of unity $q = e^{2\pi i r/s}$, the series has every such point as a singularity, so evaluation fails in the base system.

Zwegers' completion [**?**] adds a non-holomorphic correction term $\widehat{\mu}(u, v; \tau)$ to produce a *harmonic Maass form*. The clause-level reading: this is an extension $\mathcal{E}_1^{\text{mock}} = \mathcal{E}_0^{\text{mock}} + \Delta_{\text{Zwegers}}$ where $\Delta_{\text{Zwegers}}$ consists of evaluation clauses at the boundary $|q| = 1$ (and beyond, via the harmonic completion).

The parallel with RMT is exact:

(i) Both are non-neutral completions: the values added by $\Delta_{\text{Zwegers}}$ and $\Delta_{\text{ac}}$ are not identity elements.

(ii) Both are non-conservative: new value-judgements are created at previously failing sites.

(iii) Both carry a non-uniqueness cost: the Zwegers completion is the "canonical" one that restores modular symmetry, just as analytic continuation is the canonical extension that restores holomorphicity—but in both cases, different completions are logically possible and produce different values at the off-support inputs.

**Remark 18.21** (Historical asymmetry)**.** Ramanujan conjectured the existence of a completion that would restore near-modular behaviour for the mock thetas, without constructing it explicitly; Zwegers provided the completion roughly seventy years later. Under the clause-level reading, Ramanujan was claiming that a consistent $\Delta_{\text{Zwegers}}$ exists, i.e. that a certain completion *can* be formed without introducing contradictions. That this claim turned out to be correct—and that the specific completion has deep connections to harmonic analysis—is a mathematical discovery about the completions consistent with Ramanujan's data, not a discovery about the original base system.

### 18.7 Summary: clause-level taxonomy of completion in RMT and related results

Table 1 summarises the completion moves identified in this section and locates each within the taxonomy of Definition 5.1.

The key distinction is the final column. Zero-weight (neutral) completions are eliminable by depadding; their value content is entirely contained in the base system after support restriction. Full-weight (non-neutral) completions introduce genuinely new content that cannot be recovered from $\mathcal{E}_0$ alone. Ramanujan's Master Theorem, mock theta functions, and all analytic-continuation applications lie in the latter class.

This does not impugn those results; it precisely locates the epistemological weight they carry. Any identity proved via RMT is an identity *in the completed system* $\mathcal{E}_1$, and its validity is contingent on the availability of the completion clauses at the relevant parameter values. Verifying a Ramanujan identity of this type is therefore a two-part task:

(i) verify that the required completion clauses are available (i.e. that the analytic continuation exists and is non-singular at the relevant points), and

(ii) verify that the completed-system derivation is correct.

The first part is not automatic and is precisely what the editorial tradition around Ramanujan—from Hardy through Berndt—has supplied. The clause-level framework makes this supply explicit.

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| **Result / context** | **Base support $S$** | **Completion type** | **Neutral?** |
|---|---|---|---|
| RMT (generic $\phi$) | $\mathbb{N}$ | analytic cont. | No |
| $\phi(k) = 1$ (Gamma fn.) | $\mathbb{N}$ | trivial (constant) | N/A |
| $\phi(k) = \binom{2k}{k}$ | $\mathbb{N}$ | composed (Gamma-based) | No |
| $\phi(k) = \Gamma(\alpha + k)/\Gamma(\alpha)$ | $\mathbb{N}$ | analytic cont.; pole obstruction possible | No |
| Multiplicative $\phi = \sigma_\alpha$ | $\mathbb{N}^+$ | Dirichlet series cont. | No |
| Partition boundary ($p(n) < 0$) | $\mathbb{N}$ | out-of-range clause | Yes (adds 0) |
| Coefficient padding ($q$-series) | sparse $S$ | missing-coeff. clause | Yes (adds 0) |
| Mock theta (Zwegers) | $\lvert q\rvert < 1$ | harmonic completion | No |

Table 1: Clause-level taxonomy of completion moves discussed in this paper. "Neutral" means the filler value is the identity element for the surrounding aggregation, enabling depadding normalization (Theorem 6.18).

## Use of AI tools (methods/disclosure statement)

During the preparation of this manuscript, the author used a large language model (LLM) system (e.g. OpenAI *ChatGPT*; include model/version and access date if required by the venue) as an editorial and diagnostic aid. In particular, the LLM was used to (i) proofread and revise prose for clarity, (ii) assist with LaTeX refactoring and internal consistency (e.g. duplicated labels/cross-references), and (iii) flag potential logical gaps or misstatements in definitions, propositions, and proof sketches and to propose candidate corrections.

All LLM-suggested edits and mathematical/logical modifications were reviewed, checked, and either accepted, revised, or rejected by the author. The author remains solely responsible for the correctness of the mathematics, the interpretation, and the final written form. The LLM was not used to generate new experimental data, to fabricate citations, or to replace primary verification of claims; citations and substantive mathematical statements were selected and validated by the author. No confidential or proprietary information was provided to the LLM.

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